SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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	7	20	76

TOPIC Errata and corrigenda to memo "Commentary on Scott's function spaces"

REFERENCE Memo Hofmis 7-7-76

The expansion of the GK-lemma cited as Lemma A in the memo 7-7-76 contains an error. Condition (3) in part II of Lemma A should read as follows:

(3) $T \in \underline{CL}$ and $<<_T = <<_L | (T \times T)$

As a consequence we ask that the following changes be made in the memo, in addition to the one above:

page 5a (CH.III) in the last line above condition (§) replace "iff" by "if", and add after condition (§) the phrase

"and that (§) is in fact equivalent to the assertion that ker(S) is a continuous lattice and that the "way below relation" of $[S \to S]$ induces that of ker(S)."

pages 6,7, and notably Corollary 24: The "way below relation" referred to on these pages is always that of $[S \to S]$, and not that of ker(S) (if such a relation on ker(S) should exist).

page 8, Theorem I: replace condition (1) by

(1) ker (S) $\in \underline{CL}$, and for $f,g \in$ ker (S) one has $f <<_{ker(S)} g$ iff $f <<_{[S\to S]} g$.

The upshot of this is that the previous memo only gives a sufficient condition for ker(S) to be a $\underline{CL}$-object. We shall see shortly that Theorem I remains valid as first stated, but this relies on the particular nature of the kernel map from $[S \to S]$ to ker (S). The following example shows that, in general, the old condition (3) of Lemma A does not imply the other conditions:

Example: Let $S = [0,1]$, the unit interval, and $f \in (S \to S)$ by $f(x) = 1$ if $x = 1$, and $f(x) = 0$ otherwise. Then f satisfies (i), (ii), and (iii) of Lemma A, but f does not satisfy (iv); however, $f(S) = \{0,1\} \in \underline{CL}$. General principle: in a $\underline{CL}$-object S, choose an open prime ideal I, and a closed subsemilattice T of S which is a retract of I, and define $k \in (S \to S)$ to be the identity on $S \setminus I$, and the retraction of I onto T on I. For instance, let $S = I \times I$, $J = \{(x,y) : x < 1\}$ and $T = \{(x,0) : x < 1\}$. Then define $k \in (S \to S)$ by $k(x,y) = (x,y)$ if $x = 1$, and $k(x,y) = (x,0)$ if $x < 1$.

We now return to the situation of $S \in \underline{CL}$ and ker(S).

Definition 1. If $x \leq y \in S$, define $[x \leftarrow y] : S \to S$ by $[x \leftarrow y](z) = z$ if $z \notin \downarrow y$, and $[x \leftarrow y](z) = zx$ if $zy = z$.

Lemma 2. For $x \leq y \in S$, $[x \leftarrow y] \in \ker(S)$.

proof. Let $a \leq b \in S$. If $a \notin \downarrow y$, then $[x \leftarrow y](a) = a \leq b = [x \leftarrow y](b)$, while, $ay = a$ implies $[x \leftarrow y](a) = ax \leq bx \leq [x \leftarrow y](b)$. It is clear that $[x \leftarrow y] \leq 1_S$, and that $[x \leftarrow y]^2 = [x \leftarrow y]$. Finally, if $\mathcal{D} \uparrow \subseteq S$ and $z = \sup \mathcal{D}$, then $zy = z$ implies $\mathcal{D} \subseteq \downarrow y$, so that $[x \leftarrow y](z) = zx = (\sup \mathcal{D})x = (\lim \mathcal{D})x = \lim_{\mathcal{D}} dx = \sup_{\mathcal{D}} dx$. $= \sup [x \leftarrow y](\mathcal{D})$. []

Proposition 3. If $k, h \in \ker(S)$ and $k \ll h$, then $k(x) \ll h(x)$ for each $x \in S$.

Proof. Suppose that $x_0 \in S$ with $k(x_0) \not\ll h(x_0)$, and assume that $k \leq h$. Then, $k(S) \subseteq h(S)$ (see Proposition 25 of reference), and $h(S) \in \underline{CL}$. Thus there is $\mathcal{D} \uparrow \subseteq h(S)$ with $h(x_0) = \sup \mathcal{D}$ but $k(x_0) \not\leq d$ for all $d \in \mathcal{D}$. Now, Lemma 1 implies $[d \leftarrow h(x_0)] \in \ker(h(S))$ for all $d \in \mathcal{D}$. Moreover, if $x \in h(S)$ and $x \notin \downarrow h(x_0)$, then $[d \leftarrow h(x_0)](x) = x$ for all $d \in \mathcal{D}$, while, if $xh(x_0) = x$, then $h(x_0) = \sup \mathcal{D}$ implies $\sup [d \leftarrow h(x_0)](x) = \sup_{\mathcal{D}} xd = x$ as $h(S)$ is lower continuous. Thus, we have $\sup [d \leftarrow h(x_0)] = 1_{h(S)}$, and we can conclude that $\sup ([d \leftarrow h(x_0)] \circ h) = h$ in $\ker(S)$. Finally, for all $d \in \mathcal{D}$, $([d \leftarrow h(x_0)] \circ h)(x_0) = [d \leftarrow h(x_0)](h(x_0)) = dh(x_0)$ $= d$, while $k(x_0) \not\leq d$, and so $k \not\leq [d \leftarrow h(x_0)] \circ h$. This shows that $k \not\ll h$, and the desired result follows by contraposition. []

Corollary 4. For $k \in \ker(S)$, $k \ll 1$ implies that $k(S) \subseteq K(S)$. Consequently, $k \in K(\ker(S))$ implies $k(S) \subseteq K(S)$.

Proof. $k \ll 1$ implies $k(x) \ll 1(x) = x$ for all x in S. In particular, if $x \in k(S)$, then $k(x) \ll x = k(x)$ as $k^2 = k$. Thus $k(S) \subseteq K(S)$, and the result follows. []

Lemma 5. If $k \in \ker(S)$, then $h \mapsto hk : \ker(S) \longrightarrow \ker(k(S))$ is a surmorphism. Moreover, $\ker(S) \in \underline{CL}$ implies this map is continuous, so that $\ker(k(S)) \in \underline{CL}$.

Proof. Clearly, all we need show is that the image of the translation map is in fact $\ker(k(S))$, since the rest is well-known. Now, the map is a surmorphism onto $\ker(S)k = \{ h \in \ker(S) : h \leq k \}$. However, $h \leq k$ iff $h(S) \subseteq k(S)$ (again see Proposition 25 of the reference), and clearly then, $h \leq k$ implies that $h|k(S) \in \ker(k(S))$. Conversely, if $h' \in \ker(k(S))$, then it follows routinely that $h' \circ k \in \ker(S)$, and for x in S, $h'(k(x))k(x) = h'(k(x))$ as $h' \leq 1_{k(S)}$. []

Theorem II. Let $S \in \underline{CL}$. If $\ker(S) \in \underline{CL}$, then S is a dimensionally stable $\underline{Z}$-object.

Proof. Let $g:S \to S'$ be a surmorphism of S onto an S' in $\underline{CS}$. If $d:S' \to S$ is the right adjoint of g, then $f = dg \in \ker(S)$ as in the proof of (1') implies (2) of Theorem I of the reference. Hence $\ker(S') \in \underline{CL}$ by Lemma 5, and so $1_{S'} = \sup\{h \in \ker(S') : h \ll 1_{S'}\}$. But, $h \ll 1_{S'}$ implies $h(S') \subseteq K(S')$, and so, if $x \in S'$, then $x = 1_{S'}(x) = \sup_{h \ll 1_{S'}} h(x) \leq \sup(\downarrow x \cap K(S')) \leq x$. Thus $K(S')$ is dense in S', whence $S' \in \underline{Z}$. Thus, every surmorphic image of S is in $\underline{Z}$, and this shows that S is a stable $\underline{Z}$-object. []

Corollary. For $S \in \underline{CL}$, the following are equivalent:

1. $\ker(S) \in \underline{CL}$.

2. S is a dimensionally stable $\underline{Z}$- object.

Proof. Theorem II shows 1. implies 2, while Theorem I of the reference shows the converse. []

Note further that Proposition 29 remains valid to show that if $\ker(S) \in \underline{CL}$, then $\ker(S)$ is itself a dimensionally stable $\underline{Z}$-object.