SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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	8	4	76

TOPIC Errata and corrigenda to memo "Commentary on Scott's function spaces"

REFERENCES
1. Memo Hofmis 7-7-76
2. Memo Keimel 8-1-76

The expansion of the GK-Lemma cited as Lemma A in reference 1 contains an error. Conditions (2) and (3) of Part II of Lemma A should be combined to read as follows:

(2) $T \in \underline{CL}$ and $\ll_T = \ll_L \mid (T \times T)$.

As a consequence, we ask that the following changes be made in the memo, in addition to the one above:

page 5a(CH.III) in the last line above condition (§) replace "iff" by "if", and after condition (§), add the phrase:

"and that (§) is in fact equivalent to the assertion that ker(S) is a continuous lattice and that the "way below relation" of $[S \to S]$ induces that of ker(S)."

pages 6,7, and notably Corollary 24: The "way below relation" referred to on these pages is always that of $[S \to S]$, and not that of ker(S)(if such a relation on ker(S) should exist).

page 8,Theorem I: replace condition (1) by:

(1) $\ker(S) \in \underline{CL}$, and for $f,g \in \ker(S)$, one has $f \ll_{\ker(S)} g$ iff $f \ll_{[S \to S]} g$.

The upshot of this is that the previous memo only gives a sufficient condition for ker(S) to be a CL-object. We shall see shortly that Theorem I remains valid as first stated, but this relies on the particular nature of the kernel map from $[S \to S]$ to ker(S). The following examples show that neither of the conditions (2) or(3) of the original Lemma A are equivalent to the other conditions:

Example 1. Let $S = I \times I$, the unit square, and define $f \in (S \to S)$ by $f(x,y) = (x,y)$ if $x = 1$ or $y = 1$, and $f(x,y) = (0,0)$ otherwise. Then, it is readily verified that f satisfies (i), (ii), and (iii) of Lemma A, but clearly f does not satisfy (iv). However, if $T = f(S)$, then it is true that $\ll_T = \ll_S \mid (T \times T)$. The problem here is that $T \notin \underline{CL}$, since multiplication on T is not separately continuous. This example shows that condition (2) of the old Lemma A does not imply condition (iv) of the old Lemma A.

Example 2. This time, let $S = [0,1]$, the unit interval, and $f \in (S \to S)$ by $f(x) = 1$ if $x = 1$, and $f(x) = 0$ otherwise. Then, it is clear that f satisfies (i), (ii), and (iii) of Lemma A, but, again, f does not satisfy (iv); however, $f(S) = \{0,1\} \in \underline{CL}$. As a general principle, in a CL-object S, choose an open prime ideal J, and a closed subsemilattice T of S which is a retract of J, and define $f \in (S \to S)$ to be the identity on $S\backslash J$, and the retraction of J onto T on J. For instance, let $S = I \times I$, $J = \{(x,y) : x < 1\}$, and $T = \{(x,0) : x < 1\}$. Then define $k \in (S \twoheadrightarrow S)$ by $k(x,y) = (x,y)$ if $x = 1$, and $k(x,y) = (x,0)$ if $x < 1$. This example shows that the condition (3) does not imply the other conditions of part II in the old Lemma A.

A proof of the new Lemma A(i.e. with the conditions (2) and (3) combined) is in ref.[2]

We now return to the situation of $S \in \underline{CL}$ and the study of ker(S).

Definition 1. If $x \leq y \in S$, define $[x \leftarrow y] : S \longrightarrow S$ by $[x \leftarrow y](z) = z$ if $z \notin \downarrow y$, and $[x \leftarrow y](z) = zx$ if $zy = z$.

Lemma 2. For $x \leq y \in S$, $[x \leftarrow y] \in \ker(S)$.

proof. Let $a \leq b \in S$. If $a \notin \downarrow y$, then $[x \leftarrow y](a) = a \leq b = [x \leftarrow y](b)$, while, $ay = a$ implies $[x \leftarrow y](a) = ax \leq bx \leq [x \leftarrow y](b)$. It is clear that $[x \leftarrow y] \leq 1_S$, and that $[x \leftarrow y]^2 = [x \leftarrow y]$. Finally, if $\mathcal{D} \uparrow \subseteq S$ and $z = \sup \mathcal{D}$, then $zy = z$ implies $\mathcal{D} \subseteq \downarrow y$, so that $[x \leftarrow y](z) = zx = (\sup \mathcal{D})x = (\lim \mathcal{D})x = \lim\limits_{\mathcal{D}} dx = \sup\limits_{\mathcal{D}} dx = \sup\, [x \leftarrow y](\mathcal{D})$. $\square$

Proposition 3. If $k,h \in \ker(S)$ and $k \underset{ker(S)}{\ll} h$, then $k(x) \underset{S}{\ll} h(x)$ for each $x \in S$.

Since the inclusion $h(S) \hookrightarrow S$ is the adjoint of h, we have $k(x_0) \not\ll_{h(S)} h(x_0)$.

Proof. Suppose that $x_0 \in S$ with $k(x_0) \underset{S}{\not\ll} h(x_0)$, and assume that $k \leq h$. Then, $k(S) \subseteq h(S)$ (see Proposition 25 of reference), and $h(S) \in \underline{CL}$. Thus there is $\mathcal{D} \uparrow \subseteq h(S)$ with $h(x_0) = \sup \mathcal{D}$ but $k(x_0) \not\leq d$ for all $d \in \mathcal{D}$. Now, Lemma 1 implies $[d \leftarrow h(x_0)] \in \ker(h(S))$ for all $d \in \mathcal{D}$. Moreover, if $x \in h(S)$ and $x \notin \downarrow h(x_0)$, then $[d \leftarrow h(x_0)](x) = x$ for all $d \in \mathcal{D}$, while, if $xh(x_0) = x$, then $h(x_0) = \sup \mathcal{D}$ implies $\sup\, [d \leftarrow h(x_0)](x) = \sup\limits_{\mathcal{D}} xd = x$ as $h(S)$ is lower continuous. Thus, we have $\sup\, [d \leftarrow h(x_0)] = 1_{h(S)}$, and we can conclude that $\sup\, ([d \leftarrow h(x_0)] \circ h) = h$ in ker(S). Finally, for all $d \in \mathcal{D}$, $([d \leftarrow h(x_0)] \circ h)(x_0) = [d \leftarrow h(x_0)](h(x_0)) = dh(x_0) = d$, while $k(x_0) \not\leq d$, and so $k \not\leq [d \leftarrow h(x_0)] \circ h$. This shows that $k \not\ll h$, and the desired result follows by contraposition. $\square$

Corollary 4. For $k \in \ker(S)$, $k \underset{ker(S)}{\ll} 1$ implies that $k(S) \subseteq K(S)$. Consequently, $k \in K(\ker(S))$ implies $k(S) \subseteq K(S)$.

Proof. $k \ll 1$ implies $k(x) \ll 1(x) = x$ for all x in S. In particular, if $x \in k(S)$, then $k(x) \ll x = k(x)$ as $k^2 = k$. Thus $k(S) \subseteq K(S)$, and the result follows. $\square$

Lemma 5. If $k \in \ker(S)$, then $h \mapsto hk : \ker(S) \longrightarrow \ker(k(S))$ is a surmorphism. Moreover, $\ker(S) \in \underline{CL}$ implies this map is continuous, so that $\ker(k(S)) \in \underline{CL}$.

Proof. Clearly, all we need show is that the image of the translation map is in fact ker(k(S)), since the rest is well-known. Now, the map is a surmorphism onto $\ker(S)k = \{ h \in \ker(S) : h \leq k \}$. However, $h \leq k$ iff $h(S) \subseteq k(S)$ (again see Proposition 25 of the reference), and clearly then, $h \leq k$ implies that

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Theorem II. Let $S \in \underline{CL}$. If $\ker(S) \in \underline{CL}$, then S is a dimensionally stable $\underline{Z}$-object.

Proof. Let $g : S \longrightarrow S'$ be a surmorphism of S onto an S' in $\underline{CS}$. If $d : S' \longrightarrow S$ is the right adjoint of g, then $f = dg \in \ker(S)$ as in the proof of (1') implies (2) of Theorem I of the reference. Hence $\ker(S') \underset{\simeq \ker(f(S))}{\in} \underline{CL}$ by Lemma 5, and so $1_{S'} = \sup \{ h \in \ker(S') : h \ll_{\ker(S')} 1_{S'} \}$. But, $h \ll_{\ker(S')} 1_{S'}$ implies $h(S') \subseteq K(S')$, and so, if $x \in S'$, then $x = 1_{S'}(x) = \sup_{h \ll 1_{S'}} h(x) \leq \sup (\downarrow x \cap K(S')) \leq x$. Thus $K(S')$ is dense in S', whence $S' \in \underline{Z}$. Thus, every surmorphic image of S is in $\underline{Z}$, and this shows that S is a stable $\underline{Z}$-object. $\square$

Corollary. For $S \in \underline{CL}$, the following are equivalent:

1. $\ker(S) \in \underline{CL}$.
2. S is a dimensionally stable $\underline{Z}$- object.

Proof. Theorem II shows 1. implies 2, while Theorem I of the reference shows the converse. $\square$

Note further that Proposition 29 remains valid to show that if $\ker(S) \in \underline{CL}$, then $\ker(S)$ is itself a dimensionally stable $\underline{Z}$-object.