SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME(S)	Keimel	DATE M	D	Y
		8	10	76
TOPIC	Complements to relations with the interpolation properties and continuous lattices			
REFERENCE	(1) My MEMO SCS 8-1-76, (2) SCS Memo of LAWSON 7-12-76			

8. Solution of Problem (7.2) in (1).

K.H. Hofmann has pointed out to me, that an affirmative answer to this problem can be drawn from Proposition 4 in (2). We restate the following definition from (2):

(8.1) In a complete lattice S we write $x \lll y$, if the following holds: For every updirected subset D of S such that $y \leq \sup D$, there is an element d in D such that $x \ll d$.

We note:

(8.2) If $x \ll y \ll z$, then $x \lll z$.

(8.3) If $x \lll y$, then $x \ll y$.

(8.4) $x \lll^* y$ iff $x \ll^* y$ (For $\ll^*$ see sec.5 in 1).

Proposition 4 in (2) asserts:

(8.5) In every compact semilattice, $x \lll y$ iff $x \prec y$, where $x \prec y$ is defined to mean that y is interior to the principal filter generated by x as in sec. 7.

Thus, for compact semilattices we conclude:

(8.6) If $x \ll y \ll z$, then $x \prec z$.

(8.7) $x \ll^* y$ iff $x \prec^* y$.

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The last statement shows that the left reflector $\mathcal{P}_\sqsubset \circ K : \underline{CS} \longrightarrow \underline{CL}$ constructed in sec. 7 is nothing but the restriction of the left reflector $\mathcal{P}_\sqsubset \circ W' : \underline{L} \longrightarrow \underline{CL}$ in sec.6 to the 'subcategory' $\underline{CS}$ of $\underline{L}$. For the same reason the kernel operators v_S and w_S of (6.5) and (7.11) coincide on a compact semilattice S.

9. Further comments.

Lawson's proposition (8.5) which even would deserve the name LEMMA has another interesting consequence (the equivalence of (1) and (4) generalises (1.25) of ATLAS where both S and T are supposed to be continuous):

(9.1) PROPOSITION. Let S be a compact and T a continuous lattice. Let $g : S \to T$ preserve arbitrary infima, with right adjoint $d : S \to T$. Then the following are equivalent:

(1) g preserves updirected suprema.

(2) $x \underset{T}{\ll} y \Rightarrow d(x) \underset{S}{\ll} d(y)$.

(3) $x \underset{T}{\prec} y \Rightarrow d(x) \underset{S}{\prec} d(y)$.

(4) g is continuous (topologically).

Proof. (1)$\Leftrightarrow$(2) by (6.3) and (3)$\Leftrightarrow$(4) by ATLAS (1.19). Clearly, (4)$\Rightarrow$(1).

(2) $\Rightarrow$ (3) $\quad x \underset{T}{\prec} y \iff x \underset{T}{\ll} y$ as T is a continuous lattice,

$\Rightarrow \exists a \in T \quad x \underset{T}{\ll} a \underset{T}{\ll} y$ as the interpolation prop. holds in a continuous lattice.

$\Rightarrow \quad d(x) \underset{S}{\ll} d(a) \underset{S}{\ll} d(y)$ by (2)

$\Rightarrow \quad d(x) \underset{S}{\prec} d(y)$ by (8.6)

(9.2) Let $(L, \sqsubset)$ be a $\underline{CSRIP}$-object, and $c : L \to \mathcal{P}_\sqsubset L$

$$a \mapsto c(a) = \{x \in L \mid x \sqsubset a\}.$$

For c to preserve updirected suprema it is necessary and sufficient that $x \sqsubset y \Rightarrow x \underset{L}{\ll} y$. In this case $c : L \to \mathcal{P}_\sqsubset L$ is surjective. Its right adjoint $I \mapsto \sup I : \mathcal{P}_\sqsubset L \to L$ is injective and the kernel operator

$$u_\sqsubset : a \mapsto \sup c(a) : L \to L$$

preserves updirected suprema.

Proof. Suppose that c preserves updirected suprema and let $x \sqsubset y$. In order to prove $x \ll y$ let D be an updirected subset of L with $y \lneq \sup D$. Then $x \in c(y) \subseteq c(\sup D) = \bigcup\{c(d) \; ; \; d \in D\}$, whence $x \in c(d)$, i.e. $x \sqsubset d$ for some $d \in D$. Conversely, suppose that $x \sqsubset y \Longrightarrow x \ll y$. Let D be an updirected subset of L and $a = \sup D$. Clearly, $c(a)$ contains $c(d)$ for all d in D. At the other hand, if $x \in c(a)$, then $x \sqsubset a$, whence $x \sqsubset a' \sqsubset a$ for some a'. Then $a' \ll a = \sup D$ by hypothesis. Consequently, $a' \leq d$ for some d in D. We conclude that $x \in c(d)$.

For the surjectivity of c, note that for every ideal $I \in \mathcal{I}_{\sqsubset}(L)$ we have $I = \bigcup\{c(d) \; ; \; d \in I\} = c(\sup I)$, if we assume that c preserves updirected suprema. The remaining assertions are then clear.

Now let $(L, \sqsubset)$ be a $\underline{SRIP}$-object (see sec. 3). The map $c : L \to \mathcal{I}_{\sqsubset}(L)$ defined as above preserves finite suprema if and only if the following axiom is satisfied:

(9.3) If $x \sqsubset a \vee b$, then there are $x_1 \sqsubset a$, $x_2 \sqsubset b$ such that $x \leq x_1 \vee x_2$.

By (3.17) we conclude:

(9.4) $c : L \to \mathcal{I}_{\sqsubset}(L)$ is a $\underline{SRIP}$-morphism iff (9.3) holds.

Together with (9.2) this yields:

(9.5) Let $(L, \sqsubset)$ be a $\underline{CSRIP}$-object. Then $c : L \to \mathcal{I}_{\sqsubset}(L)$ is a $\underline{CSRIP}$-morphism if and only if $(L, \sqsubset)$ satisfies (9.3) and the property "$x \sqsubset y \Rightarrow x \ll_L y$".

(9.6) Under the conditions of (9.5), $c : L \to \mathcal{I}_{\sqsubset}(L)$ not only has a right adjoint $I \mapsto \sup I : \mathcal{I}_{\sqsubset}(L) \to L$, but also a left adjoint $g : \mathcal{I}_{\sqsubset}(L) \to L$ given by

$$g(I) = \sup\{a \mid c(a) \subseteq I\} \qquad (I \in \mathcal{I}_{\sqsubset}(L)).$$
$$= \sup\{a \mid x \sqsubset a \Rightarrow x \in I\}$$
$$= \sup\{a \mid I = c(a)\}$$