SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC THE RANDOM UNIT INTERVAL .(Another example of a CL objet)

REFERENCE *D. Scott, LNM 274*
K.H. Hofmann, Zur mathematischen Theorie des Messens, Dissert. Math. 32 (1963), 1-32

The traditional mathematical model in the theory of measurement is the unit interval $I = [0,1]$ with its natural order. If a set of physical objects D is given , then a <u>process of measurement</u> is a function $D \to I$ which, in general will respect some structure of D, e.g. a (partial) quasi order . If ,to produce a concrete example, D is the set of all pencils in the department,we can compare two of them in relation to their magnitude by placing one next to another. This gives a ~~partial order~~ quasiorder and any process of measurement respecting this mode of comparison qualifies to be called measurement of length. Objects with the same value would have to be declared of equal length. Each assignment $d \mapsto x_d$ of a real number $x \in I$ to an object $d \in D$ is a measurement.

The crux is that no accurate measurements exist. Each object $d \in D$ gives rise to a rather fuzzy piece of information as what value in I should be assigned to it. In reality, what we assign to an object d is a random variable X with values in I. Recall that a random variable is given by a regular probability measure μ on I; equivalently, one may characterize its probabilistic behavior by its distribution function $F = F_X$. The relation between the distribution function and the associated measure is given in terms of

$$(1) \qquad F(r) = \mu(\downarrow r)$$

We recall that distribution functions are monotone,continuous from the right and satisfy $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$.

For random variable taking values in I this reduces to $F(x) = 0$ for $x < 0$, $F(x) = 1$ for $1 \le x$.

In the light of our model for the theory of measurement we define a quasiorder on the class of random variables (attaining their values in I) by setting

(2) $X \tilde{<} Y$ iff $F_X \geq F_Y$.

This says, if you consider the definition for a moment, that Y is more likely to have larger values than X. Evidently we have

(3) $X \tilde{<} Y \Rightarrow E(X) \leq E(Y)$ (with $E(X)$ = expectation of X $= \int x\, dF_X(x)$)

which is quite reasonable. In the embellished model , if the measurement X is associated to d, we still end up obtaining a real number E(X) which represent the most probable value of the measurement. The assignment $d \longmapsto X \longmapsto E(X)$ will still reflect order structures.

The following definition now introduces the random unit interval:

DEFINITION. The random unit interval $\mathbb{I}$ is the set of all distribution functions $F: \mathbb{R} \longrightarrow [0,1] = I$ with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $1 \leq x$, equipped with the partial order ~~of functions~~ $\lhd$ given by $G \lhd F$ iff

$F \leq G$ iff $F(x) \leq G(x)$ for all $x \in \mathbb{R}$;

and equipped with the weak topology: $\lim F_j = F$ iff $\int f dF_j \longrightarrow \int f dF$ for all continuous f (on $\mathbb{R}$ or, if one wishes, on I). □

Most texts in probability will tell you that we have the following

FACT. If F_n is a sequence in $\mathbb{I}$, then $F = \lim F_n$ (in the weak topology) iff

(4) $F(x) = \lim F_n(x)$ for all points x of continuity of F.□

Notice that $\mathbb{I}$ is separable metric as the space of probability measures of the compact separable metric space I (with the weak topology); thus sequences suffice to describe the topology.

The purpose of this memo is to point out the following observation:

THEOREM. Both the random unit interval $\mathbb{I}$ and its opposite $\mathbb{I}^{op}$ are CL - objects and $\mathbb{I}$ is a topological lattice in the weak topology. In particular, the CL - topologies of $\mathbb{I}$ and $\mathbb{I}^{op}$ agree with the weak topology

More information will be given afterwards; the proof of the theorem is divided into smaller steps. For any monotone function $f: \mathbb{R} \longrightarrow [0,1]$ we write $f_*(x) = \lim f(y)$, $y \longrightarrow x$, $y < x$; note $f_*(0) = 0$.

LEMMA A. Π is a topological semilattice relative to the pointwise min operation.

Proof. We set $(F\wedge G)(x) = F(x)\wedge G(x)$. Since Π is compact, by Lawson's theorem on the joint continuity of compact semitopological semilattices, we only need to show separate continuity. It suffices to operate with sequences. So let $F = \lim F_n$; we must show $F\wedge G = \lim F_n\wedge G$. For this purpose we take $x \in \mathbb{R}$ and assume that x is a point of continuity of $F\wedge G$; by the FACT we must verify $(F\wedge G)(x) = \lim(F_n\wedge G)(x)$. Let $H \in \{F,G\}$ be such that $H(x)=F(x)\wedge G(x)$. We claim that H is continuous in x: Indeed (using that $F\wedge G$ is continuous in x and H is nondecreasing) $(F\wedge G)(x) = (F\wedge G)_*(x) \leq H_*(x) \leq H(x) = (F\wedge G)(x)$. Thus $H_*(x) = H(x)$, which proves the claim since H is continuous from the right. We proceed by case distinction: Case i: $H = F$. Then $F(x) = \lim F_n(x)$ by FACT, and so $(F\wedge G)(x) = F(x)\wedge G(x) = \lim F_n(x)\wedge G(x) = \lim (F_n\wedge G)(x)$. Case ii: $H = G$. Let $r < G(x) = F(x)\wedge G(x)$. Since G is continuous in x by the claim there is an $a < x$ such that $a < y \leq x$ implies $r < G(y)$ and all but ~~at~~ countably many of these y must be points of continuity of F, whence $F(y) = \lim F_n(y)$ for these y by FACT. Thus $(F\wedge G)(y) = \lim (F_n\wedge G)(y)$ for all such y. Since $F\wedge G$ is continuous in x there must be a point of continuity of F with $a < y \leq x$ such that $\lim F_n\wedge G)(y) > r$./By monotonicity, $\underline{\lim}(F_n\wedge G)(x) > r$. But $\overline{\lim}\,(F_n\wedge G)(x) \leq G(x) = (F\wedge G)(x)$. Since r was arbitrary with $r < G(x)$ we conclude $\lim (F_n\wedge G)(x) = (F\wedge G)(x)$.$\square$

LEMMA A'. Π is a topological semilattice relative to the pointwise max operation.

Proof. Analogous.$\square$

This shows that Π is a topological lattice relative to the weak topology. We now turn to the lattice theoretical aspects.

LEMMA B. The function $f \longmapsto \tilde{f}$ defined by $\tilde{f}(x) = 1 - f_*(1-x)$ gives a lattice isomorphism $\Pi \longrightarrow \Pi^{op}$ which is its own inverse.

Proof. If $f \in \Pi$, then $\tilde{f} \in \Pi$ (straightforward verification); also $\tilde{\tilde{f}} = f$ (immediate). If $f \leq g$ in Π, then $\tilde{g} \leq \tilde{f}$ (clear).$\square$

LEMMA C. Let $F, G \in \amalg$. Then the following statements are equivalen

(1) For all $x < 1$ with $0 < G_*(x)$ one has $F(x) < G_*(x)$.

(2) $F \ll G$.

Proof. (1) =>(2) : Let H_n be an ascending sequence in $\amalg$ with $H = \sup H_n$. Then also $H = \lim H_n$ by Lemma A. Suppose that $G \leq H$. Let $0 \leq x < 1$. Since $F(x) < G_*(x)$ we find an $r_x > F(x)$ and real numbers u_x, v_x with $u_x < x < v_x$ such that $y \in [u_x, v_x]$ implies $F(y) < r_x < G(y)$. Let $c_x \in [u_x, x[$ be an arbitrary point of continuity of H. Then $H(c_x) = \lim H_n(c_x) \geq G(c_x) > r_x$. Let n_x be a natural number with $H_{n_x}(c_x) > r_x$. Then for all $y \in [c_x, v$ we have $H_{n_x}(Y) \geq H_{n_x}(c_x) > r_x > F(y)$. We cover I by finitely many intervals $]c_{x_j}, v_{x_j}[$, $j=1,\dots,m$ and set $n = \max\{n_{x_1}, \dots, n_{x_m}\}$. Then $H_n \geq F$. (Note: This works with any up-directed net just as well as with a sequence.)

not (1) => not (2): Suppose that we have an $x < 1$ with $0 < G_*(x) \leq F(x$ Let N be a natural number with $N \geq 1/G_*(x)$. For all $n \geq N$ we define

$$H_n(y) = \begin{cases} (G(y) - \frac{1}{n}) \vee 0 & \text{for } y < x \\ G_*(x) - \frac{1}{n} & \text{for } x \leq y < (x + \frac{1}{n}) \wedge 1 \\ G(y) & \text{for } (x + \frac{1}{n}) \wedge 1 \leq y \end{cases} .$$

Then $G = \lim H_n$ in $\amalg$ and H_n is increasing. But $H_n(x) < G_*(x) \leq F(x)$ Thus $F \leq H_n$ fails for all $n \geq N$; thus $F \ll G$ fails.□

The zero element 0 of $\amalg$ is way below every element in $\amalg$,in particular it is way below F with $F(x) = 0$ for $x < 1/2$ and $= 1$ for $1/2 \leq x$, but F is not way above 0 (where way above means way below in $\amalg^{op}$). However, if F and G are such that $0 < F(x), G(x) < 1$ for $0 < x < 1$, then $F \ll G$ iff G is way above F.

LEMMA D. For $G \in \mathrm{II}$ define $\underline{G} = \sup \Downarrow G$ in II. Then

$\underline{G} = \sup \{ F \in \mathrm{II} \mid F \le G_* \}$.

Proof. Since $F \ll G$ implies $F \le G_*$ by Lemma C we have $\underline{G} \le \sup \{F \mid F \le G_*\}$. However, if $F \le G_*$ then $(F - \frac{1}{n}) \vee 0 \ll G$ by Lemma C. Hence $(F \vee \frac{1}{n}) \vee 0 \le \underline{G}$ by definition of $\underline{G}$. But $F = \sup_n (F \vee \frac{1}{n}) \vee 0$, whence $F \le \underline{G}$. Thus $\sup\{F \mid F \le G_*\} \le \underline{G}$. $\square$

LEMMA E. For all $G \in \mathrm{II}$ we have $\underline{G} = G$.

Proof. Suppose not, then there is an x with $\underline{G}(x) < G(x)$. Let r be an arbitrary element in $]\underline{G}(x), G(x)[$ Define $H_r \in \mathrm{II}$ by

$$H_r(y) = \begin{cases} 0 & \text{for } y < x + \frac{d}{2} \\ r & \text{for } x + \frac{d}{2} \le y, \end{cases}$$

where $d = d(r)$ is determined so that $x \le y < x + d$ implies $\underline{G}(y) < r$. Then we note that $H_r(y) \le G_*(y)$ for $y < x + \frac{d}{2}$ trivially; since $G(x) \le G_*(y)$ for all $x < y$ we conclude $H_r(y) = r < G(x) \le G_*(y)$ for $x + \frac{d}{2} \le y$, so $H_r \le G_*$. Thus $H_r \le \underline{G}$. But if $y \in]\frac{d}{2}, d[$, then $\underline{G}(y) < r = H_r(y)$ and this is a contradiction.

LEMMA E is precisely the assertion $\mathrm{II} \in \underline{CL}$. Since Lemma B says that $\mathrm{II}^{op} \cong \mathrm{II}$ as a lattice then also $\mathrm{II}^{op} \in \underline{CL}$. By the uniqueness of the $\underline{CL}$ – topology, Lemma A allows us to conclude that the $\underline{CL}$ topologies of II and II^{op} agree with the weak topology.

The proof of the THEOREM is finished.

PROPOSITION. Every non-degenerate interval in II contains an interval $[A,B] \cong \mathrm{II}$.

Proof. Every non-degenerate interval contains an interval $[F,G]$ with $F(x) < G(x)$ for some x. Fix r,d so that $F(x) < r < G(x)$ and $F(y) < r$ for $x \le y \le x + d$, $0 < d$. Define

$$A(y) = \left\{ \begin{array}{lll} F(y) & \text{for } y < x, & F(y) \\ r & \text{for } x \le y < x + d/2, & G(x) \\ G(x) & \text{for } x + d/2 \le y < d + x, & G(x) \\ G(y) & \text{for } x + d \le x, & G(y) \end{array} \right\} = B(y). \quad \square$$

As a consequence of This Proposition, we know that $\mathbb{I}$ is certainly not isomorphic to I^X for any X, since I^X contains intervals which are isomorphic to I.

In fact we indicate that the following is true ,too:

PROPOSITION. Every element ~~af~~ F of $\mathbb{I}$ has arbitrarily small neighborhoods which are isomorphic to $\mathbb{I}$, provided that $F \ll 1$.

Indication of proof. F has small neighborhoods of the form $[A,B]$ with $A \ll F \ll B$. By the interpolation property there are A',B' with $A \ll A' \ll F \ll B' \ll B$. Every G in $\mathbb{I}$ is the sup in $\mathbb{I}$ of the set of all continuous $H \in \mathbb{I}$ with $H \leq \mathbb{I}$. Hence there is a continuous A'' with $A \leq A'' \leq A'$; make sure that $0 < B(x) < 1$ for $0<x<1$; likewise for B'; so $F \ll B' \ll B$ are also way above relations,and the dual arguments apply to give a continuous B'' with $B' \leq B'' \leq B$. Show that $[A'' , B''] \cong \mathbb{I}$. □

Probably $F \ll 1$ has little to do with this, so that in fact all elements of $\mathbb{I}$ very likely have small neighborhoods isomorphic to $\mathbb{I}$.

~~PROPOSITION . $\mathbb{I}$ is distributive.~~
~~Proof. $\mathbb{I}$ is a sublattice of $I^{\mathbb{R}}$ which is distributive.~~ □

Proposition. $\mathbb{I}$ is closed in $I^{\mathbb{R}}$ under finite sups and infs, hence is a distributive lattice. □

PROPOSITION. $\mathbb{I}$ contains the cube $I^{\mathbb{N}}$.

Proof. The function $\varphi: I^{\mathbb{N}} \longrightarrow \mathbb{I}$ given by $\varphi(a_1,a_2,\ldots)(x)$
$= \sum \{a_n/2^n : 1 - \frac{1}{n} \leq x , n=1,2,\ldots\}$ if $x < 1$, $= 1$ if $x = 1$
is a continuous lattice embedding mapping zero to zero. □

The coproduct in CL $^{\mathbb{N}}I$ is not separable metric (since it contains the coproduct $^{\mathbb{N}}2$ = space of closed subsets of $\beta\mathbb{N}$ under $\cup$. Hence $\mathbb{I} \not\cong {}^{\mathbb{N}}2$. There do not seem to be any particularly concrete morphisms $\mathbb{I} \longrightarrow I$; the memo on strict chains (SCS 4-19-76) shows how to produce such mopphisms,since via Lemma A it is not hard to recognize strict chains in $\mathbb{I}$.

One other remark: The function $f \longmapsto f_*$ maps the set of right continuous functions bijectively and under preservation of the order onto the set of left continuous ones. Recall that according to Scott we denote the set of all left continuous functions $I \longrightarrow I$ by $[I\to I]$. Hence

REMARK. The function $f \longmapsto f_*: \mathbb{I} \longrightarrow [I\to I]$ is an isomorphism onto the ~~XXXXXX~~ CL-~~XXX~~object ~~XX~~ $[I\to I]_0 = \{f \in [I\to I]: f(0) = 0\}$

~~Wx~~ In using the result that $[I\to I] \in$ CL first proved by Scott in LNM 274 and then by different methods by Hofmis SCS 7-7-76 and again by Scott in SCS 8-23-76 we could have used the Remark in a portion of the proof of the Theorem, namely, that portion whic establishes that $\mathbb{I}$ is a CL-object.

It appears, perhaps from hindsight, that an equivalent approach to $\mathbb{I}$ would have been more compatible with semilattice theory, even thought it would be less compatible with classical probability theory. Indeed we based our discussion on the classical cumulati distribution functions F which are right continuous non-decreasin with $F(1)= 1$, $F(x)= 0$ for $x < 0$. They were introduced from the probability measures μ via $F_\mu(s) = \mu(\downarrow r)$ (see (1) on p.1).

A completely equivalent theory (for the unit interval) results if we associate with each μ ~~exP(I)~~ the function $s \longmapsto \mu(\uparrow r)$.

We make the general observation:

OBSERVATION. Let S be a compact semilattice ($S \in$ CS). If $s_j \to s$ is a convergent net, up-directed, then

$$\bigcap \uparrow s_j = \uparrow s.$$

Hence, if S is first countable and μ a probability measure, the function $s \longmapsto 1-\mu(\uparrow s)$ preserves up directed limits.

Let us denote the space of all probability measures of S in the weak (= vague) topology by $P(S)$, where S is any compact semilattice For $\mu \in P(S)$ set $\varphi_\mu(s) = 1-\mu(\uparrow s)$. We have observed:

PROPOSITION . Let $S \in$ CS be first countable. Then the assignment $\mu \longmapsto \varphi_\mu$, $\varphi_\mu(s) = 1-\mu(\uparrow s)$ gives a function $P(S) \xrightarrow{\Phi_S} [S\to T]_0$, where $f \in [S\to T]$ is in $[S\to T]_0$ iff $f(0) = 0$.

T]

We observe:

PROPOSITION. Let $S \in \underline{CL}$ be separable metric. Then $\phi: P(S) \longrightarrow [S \to I]_o$ is injective.

Proof: If the measures μ and ν agree on all principa filters $\uparrow s$ then they agree on the Borel algebra B of sets generated by the $\uparrow s$. If U is an open filter and $u = \min U$, then there is a sequen $u_n \in U$ which decreases and converges to U(first countability is used here!). . Suppose now that $U = \operatorname{int} \uparrow u = \{v: u \ll v\}$. Then $U = \bigcup \uparrow u_n$, and in this case we have $U \in B$. We know that the $\underline{CL}$-topology has a basis of open sets of the form $\operatorname{int} \uparrow s \cap (S \setminus \uparrow t_1) \cap \ldots \cap (S \setminus \uparrow t_n)$. It follows that these sets are in B. Now every compact set has a countable neighborhood basis consisting of a finite union of basic open sets,and it is in fact the intersection of this neighborhood basis. Thus every compact set belongs to B and thus B contains all Borel sets. This proves the claim. $\square$

Our entire discussion involved about the following result, which is only rephrasing our theorem

THEOREM. $\phi_I: P(I) \longrightarrow [S \longrightarrow I]_o$ is a ~~bijection and~~ homeomorphism. This allows us to transport the $\underline{CL}$-structure of $[S \longrightarrow I]_o$ back to $P(I)$ giving us the Random Unit Interval (up to isomorphism). This is, so to speak, the link between Scott's function space theory and the analysis we discussed earlier.

Let us write $P(I) = \mathbb{I}$ from here on out.

LEMMA. Let $A,B,C \in \underline{CL}$ and let $D \subseteq$ ~~xxxxx~~ $\underline{CL}^{op}(B,A)$. Then the function $\varphi \longmapsto (\varphi \circ d)_{d \in D} : [A,C] \longrightarrow [B,C]^D$ is a $\underline{CL}$-morphism.

Proof. We must show that $\varphi \longmapsto \varphi \circ d: [A,C] \longrightarrow [B,C]$ is a $\underline{CL}$-morphism for each d. The function is clearly Scott continuous (i.e. preserves up-directed sups [pardon, Scott says we don't have to say 'up-directed sups'; indeed 'directed sups' is fine]). If (A,C) denotes the $\underline{CL}$-object of all monotone functions under pointwise operations, then $\varphi \longmapsto \varphi \circ d: (A,C) \longrightarrow (B,C)$ is clearly a $\underline{CL}$-morphism. There is, however, a natural transformation of functors $\underline{CL}$ $(\underline{CL}^{op})^{op} \longrightarrow \underline{CL}$ given by $k:(-,C) \longrightarrow [-,C]$ where $k_A(f)(a) = \sup f(\Downarrow a)$. The surjectivity of k, in view of the commuting diagram

$$\begin{tikzcd}
(A,C) \arrow[r, "{(d,C)}"] \arrow[d, "k_A"'] & (B,C) \arrow[d, "k_B"] \\
{[A,C]} \arrow[r, "{[d,C]}"'] & {[B,C)}
\end{tikzcd}$$

shows that $[d,C] = (\varphi \longmapsto \varphi \circ d)$ is a $\underline{CL}$-morphism.

[REMARK. For a discussion of k_A (exclusive of the asserted naturality) see SCS Hofmis 7-7-76, and SCS Scott 8-23-76. The naturality is a matter of verification using the following Lemma: If $d: B \longrightarrow B$ is a $\underline{CL}^{op}$-morphism then, as an updirected set, $\Downarrow d(b)$ is cofinal in $d(\Downarrow b)$, ~~xxxxxxxxxx~~ (Use preservation of << by d and preservation of sups by d together with the definition of <<.) From the Lemma it follows that $\sup \varphi(\Downarrow d(b)) = \sup \varphi d(\Downarrow b)$ for every Scott continuous φ, which is precisely the commuting of the diagram above, i.e. naturality.]

COROLLARY. Let $S \in \underline{CL}$. Then $\varphi \longmapsto (\varphi \circ d)_{d \in D}: [S \to I]_\circ \longrightarrow$ ~~xxxxx~~ $[I \longrightarrow I]_\circ^{\underline{CL}^{op}(I,S)}$ is an injective $\underline{CL}$-morphism.

Proof. After the previous Lemma, only injectivity is left. Now φ and ψ have the same image iff they agree on all $d(t)$, $t \in I$, $d \in \underline{CL}^{op}(I,S)$. But this set is dense (Memo Hofmann (4-19-76) on Strict Chains.) $\square$.

Now we define a function $h: P(I)^{\underline{CL}(S,I)} \longrightarrow [I \to I]_\circ^{\underline{CL}^{op}(I,S)}$ as follows: Let $\tilde{d}$ be the left adjoint of a $\underline{CL}^{op}$-map $d: I \longrightarrow S$; then set $h((\mu_g)_{g \in \underline{CL}(S,I)}) = (\mu_{\tilde{d}})_{d \in \underline{CL}^{op}(I,S)}$. This map is an isomorphism by what we saw earlier. Now we recall that every continuous function $f:S \longrightarrow T$ induces a continuous map $P(f) : P(S) \longrightarrow P(T)$ defined by $P(f)(\mu)(X) = \mu(f^{-1}(X))$.. This allows us to define a map $\pi: P(S) \longrightarrow P(I)^{\underline{CL}(S,I)}$ by

$$\pi(\mu) = (P(g)(\mu))_{g \in \underline{CL}(S,I)} .$$

<u>LEMMA</u>. The following diagram is commutative:

$$\begin{tikzcd}
P(S) \arrow[r, "\phi_S"] \arrow[d, "\pi"'] & {[S \longrightarrow I]_o} \arrow[d, "\varphi \mapsto (\varphi \circ d)_{d \in \underline{CL}^{op}(I,S)}"] \\
P(I)^{\underline{CL}(S,I)} \arrow[r, "h"'] & {[I \to I]_o}^{\underline{CL}^{op}(I,S)}
\end{tikzcd}$$

Proof. $h\pi(\mu) = (\varphi_{P(\tilde{d})(\mu)})_{d \in \underline{CL}^{op}(I,S)}$ and $\phi_S(\mu)(s) = 1-\mu(\uparrow s)$

so we have to show that for each $d \in \underline{CL}^{op}(I,S)$ we have

$\varphi_{P(\tilde{d})(\mu)}(s) = 1-\mu(\uparrow d(s))$ for all s. But $\varphi_{P(\tilde{d})(\mu)}(s) = 1-P(\tilde{d})(\mu)(\uparrow s) = 1-\mu(\tilde{d}^{-1}(\uparrow s))$. However, $\tilde{d}^{-1}(\uparrow s) = \uparrow d(s)$ by the theory of Galois connections (ATLAS)..□

All maps with the possible exception of ϕ_S are continuous, all are injective; it follows that ϕ_S has to be continuous (compactness argument!); all maps thus are topological embeddings. We record:

<u>PROPOSITION</u> . If S is a separable metric <u>CL</u>-object, then $\phi_S : P(S) \longrightarrow [S \longrightarrow I]_o$ is a topological embedding.□

Here comes the problem:
<u>PROBLEM</u>. What do we know about $\phi_S(P(S))$?

Is it a semilattice relative to the induced order or its opposite? Simple experimenting with point masses and their finite convex combinations shows that im ϕ will not in general be closed under the formation of finite infs or sups in $[S \longrightarrow I]_o$. Note that im ϕ might still have its own infs or sups. This would allow to equip P(S) with a semilattice or lattice structure via ϕ_S. For the moment it has only the structure given by a closed partial order.