SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC The space of lower semicontinuous functions into a CL-object Applications (part I): Copowers in CL

REFERENCE [0] Handwritten notes on discussions by Gierz, Hofmann, Keimel, Mislove at Darmstadt in June 1976.

[1] Hofmann, K.H. and J.D. Lawson, Irreducibility and generation in continuous lattices. Preprint.

[2] Hofmann, K.H. and A. Stralka, ATLAS, Diss. Math. 137 (1976), 1-54

In Darmstadt this summer I raised the question of calculating copowers in CL; we knew at the time that

$^J2 = \Gamma(\beta J)$, where $\Gamma(X)$ for a compact space X is the $\cup$-semilattice of compact subsets and where the CL-topology is the Hausdorff topology. We had no particular idea what such simple coproducts as

$\coprod\ {}^J I$ might be. Then Keimel had the insight that $^J I$ should be calculated by considering the cone with basis $\beta J$; then the closed subsets containing the vertex and being star shaped would be the elements of the desired copower with $\cup$ as operation. This turned out to be correct as we proved at the time. An explicit discussion of this approach is given in an example in [1] where this information was needed and serves a useful purpose.

We thought at the time that arbitrary copowers should be calculated in an essentially similar fashion. However, there are some technical difficulties with copowers of CL-objects which are not chai The present discussion proposes an approach which probably best accomodates these difficulties; in a philosophical way, such an ~~app~~ approach had been indicated in conversations in Darmstadt, although it was then not seriously attempted.

We actually develop a theory of function spaces of lower semicontinuous functions $f: X \longrightarrow S$, X compact, $S \in$ CL. The totality of all of these functions, which we call $LC(X,S)$ turns out to be a continuous lattice in a functorial fashion. The theory ~~of~~ around this concept is discussed in Section 1. Section 2 applies this to copowers Further applications are to be discussed later. The result on copowers is that for any CL-object S we have

$$^J S \triangleq LC(\beta X, S).$$

The coproejctions and the universal morphisms are explicitly given.

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

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$${}^{J}S \cong LC(\beta X, S).$$

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## 1. Lower semicontinuous functions.

1.1. <u>LEMMA</u>. Let $X$ be a topological space and $S \in \underline{CL}$. Let $x \in X$ and let $\mathcal{U}$ denote the filter basis of open neighborhoods of $x$ in $X$. Then the following conditions are equivalent:

(1) $\underline{\lim}\, f(x_j) \geq f(x)$ for every net $x_j$ in $X$ converging to $x$.

(2) $\uparrow f(x) \supseteq \bigcap\{ f(U)^- : U \in \mathcal{U} \}$.

(3) For each $s \ll f(x)$ there is an $U \in \mathcal{U}$ such that $f(U) \subseteq \uparrow s$.

(3') For each $s \ll f(x)$ there is a $U \in \mathcal{U}$ such that $s \ll f(u)$ for $u \in U$.

Now we denote with $G(f)$ the set $\{(x,s) : f(x) \leq s\}$. Then the following conditions are equivalent:

(I) Conditions (1)–(3) above hold for all $x \in X$.

(II) $f^{-1}(\mathrm{int} \uparrow s)$ is open for all $s \in S$.

(III) $G(f)$ is closed.

Proof. (3) => (2): For each $s \ll f(x)$ we know from (3) that $f(U) \subseteq \uparrow s$ for some $U \in \mathcal{U}$, hence $f(U)^- \subseteq \uparrow s$, and so $\bigcap_{\mathcal{U}} f(U)^- \subseteq \uparrow s$. Since $s \ll f(x)$ is arbitrary and $f(x) = \sup \Downarrow f(x)$, (2) follows.

(2) => (1). Suppose $x = \lim x_j$. Then eventually $f(x_j) \subseteq f(U)^-$ for all $U \in \mathcal{U}$. So every cluster point of $f(x_j)$ is in $\bigcap_{\mathcal{U}} f(U)^-$. Hence (1).

not (3) => not (1): Suppose there is an $s \ll f(x)$ such that for each $U \in \mathcal{U}$ we had $f(U) \nsubseteq \uparrow s$. Then there exisxts an $x_U \in U$ for each $U \in \mathcal{U}$ such that $f(x_U) \ngeq s$. Since $\uparrow f(x)$ is in the interior of $\uparrow s$, not cluster point of $f(x_U)$ is in $\uparrow f(x)$, which implies $\underline{\lim}\, f(x_U) \ngeq f(x)$.

(3') => (3) is trivial and (3) => (3') since for $s \ll f(x)$ there is an $s' \in S$ with $s \ll s' \ll f(x)$

(II) <=> (2) for all $x \in X$ : Clear in view of $\mathrm{int} \uparrow s = \{t \in S : s \ll t\}$

(1) for all $x \in X$ => (III): Suppose that $(x,s) = \lim (x_j, s_j)$ with $f(x_j) \leq s_j$. Then there is a subnet such that $\underline{\lim}\, (x_j, s_j) = \lim (x_{j(k)}, s_{j(k)})$. The validity of (1) for $x$ implies $f(x) \leq \underline{\lim}\, f(x_j) = \lim f(x_{j(k)}) \leq s$.

(III) => (1) for all $x$. Suppose $x = \lim x_j$ in $X$. Let $s$ be any cluster point of $f(x_j)$ in $S$, say $s = \lim f(x$ ... $)$. Then

$(x,s) = \lim\, (x_{j(k)}, f(x_{j(k)}))$ and obviously $(x_{j(k)}, f(x_{j(k)})) \in G(f)$. Thus (III) implies $(x,s) \in G(f)$, i.e. $f(x) \leq s$. This means $f(x) \leq \underline{\lim}\, f(x_j)$ since s was an arbitrary cluster point. $\square$

1.2. DEFINITION. A function $f\colon X \longrightarrow S$ is called *lower semicontinuous* iff the equivalent conditions (I) – (III) of 1.1 are satisfied. The set of all lower semicontinuous functions will be denoted with $LC(X,S)$.

1.3. LEMMA. Let $\mathcal{F} \subseteq S^X$, then $G(\sup \mathcal{F}) = \bigcap\{G(f) : f \in \mathcal{F}\}$.

Proof. Since $f \leq \sup \mathcal{F}$, we have $G(\sup \mathcal{F}) \subseteq g(f)$ for all $f \in \mathcal{F}$, whence $G(\sup \mathcal{F}) \subseteq \bigcap_{\mathcal{F}} g(f)$. If $(x,s) \in \bigcap_{\mathcal{F}} G(f)$ then $f(x) \leq s$ for all $f \in \mathcal{F}$, thus $(\sup \mathcal{F})(x) \leq s$, whence $(x,s) \in G(\sup \mathcal{F})$. $\square$

1.4. LEMMA. Let $f,g \in LC(X,S)$. Then $fg \in LC(X,S)$ where $(fg)(x) = f(x)g(x) = f(x) \wedge g(x)$.

Proof. Let $x \in X$ and $s \ll fg(x)$. Since $f,g \in LC(X,S)$, by 1.1.(3) there is an open neighborhood U of x in X such that $f(U) \cup g(U) \subseteq \uparrow s$. Then $s \leq f(u)g(u) = fg(u)$ for all $u \in U$, which verifies 1.1.(3) for fg. $\square$

1.5. PROPOSITION. Let X be topological space and $S \in \underline{CL}$. Then $LC(X,S)$ is a sublattice of $S^X$ containing the identity and zero, and $LC(X,S)$ is closed under the formation of arbitrary sups. In particular, $LC(X,S)$ is a complete lattice.

Proof. In view of 1.1.(III), Lemma 1.3 shows that $LC(X,S)$ is closed under arbitrary sups. Lemma 1.4 shows that $LC(X,S)$ is closed under finite infs. $\square$

REMARK. In general, $LC(X,S)$ is not closed under arbitrary infs: Let $x \in X$ be a non-isolated point in some topological Hausdorff space, $S = 2$. Then the inf of the characteristic functions $\chi_U$, $U \in \mathcal{U}$ (where $\mathcal{U}$ is the set of open neighborhoods of x) is $\chi_{\{x\}}$, which is not lower semicontinuous.

It is very convenient for the following to consider characterist functions of subsets of $X$:

1.6. <u>NOTATION</u> . If $X$ is a set and $S \in$ <u>CL</u> , the for each $Y \subseteq X$ the function $\chi_Y : X \longrightarrow S$ is defined by $\chi_Y(x) = 1$ if $x \in Y$ and $= 0$ otherwise. If $s \in S$ we identify $s$ with the constant function $X \longrightarrow S$ with value $s$ and write $s\chi_Y$ for the function taking the value $s$ on $Y$ and $0$ elsewhere. []

Note that $s\chi_U \in LC(X,S)$ for a topological space $X$ and an open subset $U \subseteq X$.

1.7. PROPOSITION. Let $X$ be a compact topological space and $S \in$ <u>CL</u>. If $f,g \in LC(X,S)$ then the following statements are equivalent:

(1) $f \ll g$ . (2) For each $x \in X$ there is an open neighborhood $U=U(x)$ of $x$ in $X$ and an $s = s(x) \in S$ such that

$$f(u) \leq s \ll g(u) \text{ for all } u \in U$$

(i.e. $f(U) \subseteq {\downarrow}s$ and $g(U) \subseteq \operatorname{int} {\uparrow}s$ )

(3) $G(g) \subseteq \operatorname{int} G(f)$.

Proof. (3) <=> (2) : (3) means that for every $x \in X$ there is a basic open set of the special form $U \times \operatorname{int} {\uparrow}s$ containing $g(x)$ and being contained in $G(f)$ .But this is precisely (2).

(2) => (1): Suppose that $h_j$ is an up-directed net in $LC(X,S)$ whose sup $h$ dominates $g$. Then for each $x$ we have $g(x) \leq h(x) = \lim h_j(x)$. Thus there is a $j = j(x)$ with $s(x) \ll h_j(x)$, and since $h_j$ is lower semicontinuous there is an open set $V=V(x) \subseteq U(x)$ such that $s \ll h_j(v)$ for all $v \in V$. By the compactness of $X$ we find finitely many $x_1,\ldots,x_n$ such that $X = V(x_1) \cup \ldots \cup V(x_n)$. Let $k$ be an index with $k \geq j(x_1),\ldots,j(x_n)$ Since $h_j$ is up-directed, we conclude that for each $x \in X$ there is an $i \in \{1,\ldots,n\}$ with $f(x) \leq s(x_i) \ll h_k(x)$

In particular $f \leq h_k$. This proves (1).

(1) $\Rightarrow$ (2): Let $\mathcal{F}(g)$ be the set of all functions $s\chi_U$ such that (i) $U$ is open in $X$, (ii) $s \ll g(x)$ for all $x \in \bar{U}$ (!!). By (i) we have $\mathcal{F}(g) \subseteq LC(X,S)$. Since $X$ is regular and 1.1.(3') applies to $g$, we know that (iii) $g = \sup \mathcal{F}(g)$ in $LC(X,S)$. Hence, by the definition of $f \ll g$ there is a finite collection $\{s_i\chi_{U_i} : i=1,\dots,n\} \subseteq \mathcal{F}(g)$ with (iv) $f \leq \sup_i s_i\chi_{U_i}$. Now let us take an arbitrarz $x \in X$. Let $I(x) = \{i : i \in \{1,\dots,n\}$ and $x \in \bar{U}_i\}$. Since $s_i \ll g(y)$ for all $y \in \bar{U}_i$ by (ii) above, $i \in I(x)$ implies $s_i \ll g(x)$. If we set $s(x) = \sup\{s_i : i \in I(x)\}$ then also (v) $s(x) \ll g(x)$ since $\downarrow g(x)$ is closed under finite sups. ~~(There is an $s' \in S$ with $s(x) \ll s' \ll g(x)$.)~~ The set $V(x) = X \setminus \bigcup\{\bar{U}_i : i \in \{1,\dots,n\} \setminus I(x)\}$ is an open neighborhood of $x$. By 1.1.(3') we find an open neighborhood $U(x) \subseteq V(x)$ such that $u \in U(x)$ implies ~~($s' \leq g(u)$, hence)~~ $s(x) \ll g(u)$. But $u \in U(x)$ implies that $u \notin U_i$ for $i \notin I(x)$ whence $f(u) \leq \sup\{s_i\chi_{U_i}(u) : i=1,\dots,n\} = \sup\{s_i\chi_{U_i}(u) : i \in I(x)\} = s(x)$. This proves condition (3). □

Note that it is possible that $I(x) = \emptyset$.

1.8. LEMMA. Let $X$ be compact and $f \in LC(X,S)$. Then

$$f = \sup\{g \in LC(X,S) : g \ll f\}.$$

Proof. As was observed earlier, $f$ is the sup of the family of all $s\chi_U \in LC(X,S)$ such that $s \in S$, $U$ is open in $X$ and $s \ll f(u)$ for all $u \in U$. (Use 1.1.(3).) But by Proposition 1.7 every such $s\chi_U$ satisfies the relation $s\chi_U \ll f$. This proves the Lemma. □

1.9. RECALL. Let $T \in$ CL and $t \in T$, and $t_j$ a net. Then the following statements are equivalent: (1) $t = \lim t_j$. (2) $t = \sup_j \inf\{t_k : j \leq k\}$. □

1.10. THEOREM. Let $X$ be a compact Hausdorff space and $S$ a CL-object. Then

(i) $LC(X,S)$ is a CL-object;

(ii) $f \ll g$ iff for each $x \in X$ there is an open set $U$ and an $s \in S$ such that $f(u) \leq s \ll g(u)$ for all $u \in U$;

(iii) if $f \in LC(X,S)$ and $f_j$ is a net in $LC(X,S)$ then

If $X$ is zero dimensional, then $f \ll g$ iff there is a locally constant function $h$ with $f(x) \leq h(x) \ll g(x)$ for all $x \in X$.

Proof. (i) follows from 1.5 and 1.8. (ii) is a portion of 1.7. (iii) follows from 1.9. If $X$ is zero dimensional, then there is a cover of $X$ by disjoint compact open sets $V_1, \ldots, V_n$ which refines the cover $\{U(x) : x \in X\}$ (notation as in 1.7 and its proof). For each $j \in \{1, \ldots, n\}$ find $x$ so that $V_j \subseteq U(x)$ and set $s_j = s(x)$. Then $v \in V_j$ implies $f(v) \leq s_j \ll g(v)$. Define $h : X \longrightarrow S$ by $h(x) = s_j$ iff $x \in V_j$. ∎

We ~~conclude the section with~~ add some remarks on the functorial properties of $(X,S) \longmapsto LC(X,S)$: Comp x CL $\longrightarrow$ CL

1.11. LEMMA. Let $\varphi : X \longrightarrow Y$ be a continuous function of compact spaces. For every $f \in LC(Y,S)$ the function $f \circ \varphi : X \longrightarrow S$ is lower semicontinuous. Let $\varphi^* : LC(Y,S) \longrightarrow LC(X,S)$ be the function defined by $\varphi^*(f) = f \circ \varphi$. Then $\varphi^* \in \underline{CL}^{op}$.

Proof. From 11.(II), $f \circ \varphi$ is lower semicontinuous if $f$ is. Thus $\varphi^*$ is well-defined. Since sups are calculated pointwise in $LC(Y,S)$ and $LC(X,S)$, clearly $\varphi^*$ preserves arbitrary sups. It remains to show that $f \ll g$ in $LC(Y,S)$ implies $\varphi^*(f) \ll \varphi^*(g)$ in $LC(X,S)$. We consider the commutative diagram

$$\begin{tikzcd} LC(Y,S) \arrow[r, "\varphi^*"] \arrow[d, "G_Y"'] & LC(X,S) \arrow[d, "G_X"] \\ \Gamma(Y \times S) \arrow[r, "A \mapsto (\varphi \times 1_S)^{-1}(A)"'] & \Gamma(X \times S) \end{tikzcd}$$

$G_Y(f) = \{(y,s) \in Y \times S : f(y) \leq s\}$,
$G_X$ defined similarly.

(Indeed $(x,s) \in (\varphi \times 1_S)^{-1}(G_Y(f))$ iff $(\varphi(x), s) \in G_Y(f)$ iff $f(\varphi(x)) \leq s$ iff $(x,s) \in G_X(\varphi^*(f))$!)

Now if M,N are compact spaces and $\psi: M \longrightarrow N$ is a continuous map Then the function $\psi': \Gamma(N) \longrightarrow \Gamma(M)$ given by $\psi'(A) = \psi^{-1}(A)$ satisfies the condition $\psi'(A) \ll \psi'(B)$, whenever $A \ll B$ where $A \ll B$ means $B \subseteq \text{int } A$; for ~~since~~ $\psi^{-1}(B) \subseteq \psi^{-1}(\text{int } A) \subseteq \text{int } \psi^{-1}(A)$ by the continuity of $\psi$. Since the ~~xxxxxx~~ maps $G_Y$ and $G_X$ are injective and preserve $\ll$, by 1.7 we conclude that $\varphi^*$ preserves $\ll$. $\square$

1.12. NOTATION. In the context of 1.11 the left adjoint of $\varphi^*$, which is given by $f \longmapsto \sup\{g \in LC(Y,S): g \circ \varphi \leq f\}$, $f \in LC(X,S$ will be denoted by $LC(\varphi,S): LC(X,S) \longrightarrow LC(Y,S)$ (somewhat contrary to the customary notation used in the case of the co-functor $C(-,Z)$.)

1.13. LEMMA. Let X be a compact space and $\pi: S \longrightarrow T$ a CL-morphism. Then $LC(X,\pi): LC(X,S) \longrightarrow LC(X,T)$, $LC(X,\pi)(f) = \pi \circ f$ is well -defined and a CL-morphism.

Proof. Let $\delta: T \longrightarrow S$ be the right adjoint of $\pi$. Then $\pi(s) \geq t$ iff $s \geq \delta(t)$ for $(s,t) \in S \times T$; hence $\pi \circ f \geq g$ iff $f \geq \delta \circ g$ for $(f,g) \in S^X \times T^X$. Now $\delta: T \longrightarrow S$ is lower semicontinuous [2, ATLAS 1.20, p.15) .Hence $\delta \circ g \in LC(X,S)$ for all $g \in LC(X,T)$. Since $\delta$ preserves sups, so does $g \longmapsto \delta \circ g$. Since $\delta$ preserves $\ll$, ~~By 1.7.(2), it~~ $g \longmapsto \delta \circ g$ also preserves the way below relation $\ll$. Hence $g \longmapsto \delta \circ g: LC(X,T) \longrightarrow LC(X,S)$ is a $CL^{op}$-morphism[2, ATLAS 1.20] Thus its left adjoint $LC(X,\pi)$ is a CL-morphism. $\square$

As a consequence of 1.11-1.13 we record:

1.14. PROPOSITION. $LC(-,-)$: Comp $\times$ CL $\longrightarrow$ CL is a functor. $\square$

Note that it is a bit curious that we have COVARIANCE in both arguments; you would normally expect contravariance in the left hand argument.

One further remark:

The map $G: LC(X,S) \longrightarrow \Gamma(X\times S)$, $G(f)=\{(x,s): f(x)\le s\}$ preserves arbitrary sups by 1.3 and the way below relation $\ll$ by 1.7. Hence it is a $CL^{op}$-morphism, for what it is worth. What is its left adjoint? Let $A\subseteq \Gamma(X\times S)$. Define $L(A):X\longrightarrow S$ by $L(A)=\sup\{f\in LC(X,S): G(f)\supseteq A\}$. According to [2, ATLAS], this is the required left adjoint. Thus

1.15. PROPOSITION. The map $L: \Gamma(X\times S)\longrightarrow LC(X,S)$ given by $L(A)(x)=\sup\{f(x): f\in LC(X,S) \text{ with } A\subseteq G(f)\}$, $x\in X$ is a surjective CL-morphism. □

1.16 LEMMA. Let $S, T\in$ CL, then any monotone Scott continuous function $f:S\longrightarrow T$ is lower-semicontinuous.

Proof. Let $x\in S$ and $t\ll f(x)$. Since $x=\sup \Downarrow x$ and $f$ preserves sups of up-directed sets we have $f(x)=\sup\{f(y): y\ll x\}$. By the definition of $\ll$ there is a $y\ll x$ with $t\le f(y)$. Let $U$ be the open set $\operatorname{int}\uparrow y$. Then $U$ is a neighborhood of $x$ and $u\in U$ implies $t\le f(y)\le f(u)$. Thus by 1.1(3) the assertion follows. □

1.17 COROLLARY. $[S\longrightarrow T]\subseteq LC(S,T)$. □

In a later memo we should discuss this inclusion further and resolve such questions as the following: Is $[S\longrightarrow T]$ closed in $LC(S,T)$? There are probably links to such matters as the random unit interval (SCS Hofmann and Liukkonen 9-1-76).

## 2. APPLICATIONS I . The copowers.

2.1. DEFINITION. Let $X$ be a compact space and $S \in \underline{CL}$ , $T \in \underline{CL}$ . A hemimorphism $F: X \times S \longrightarrow T$ is a lower semi-continuous function such that $s \longmapsto F(x,s): S \longrightarrow T$ is in $\underline{CL}$ for all $x \in X$.

For each pair $(x,s)$ we denote with $\Delta(x,s): X \longrightarrow S$ the function given by

$$\Delta(x,s)(y) = \begin{cases} s \text{ if } y = x \\ 1 \text{ otherwise} \end{cases} .$$

2.2. REMARK . $\Delta: X \times S \longrightarrow LC(X,S)$ is a hemimorphism.

Proof. We have $G(\Delta(x,s)) = (X \times \{1\}) \cup (\{x\} \times \uparrow s)$. Clearly $(x,s) \longmapsto G(\Delta(x,s)) : X \times S \longrightarrow \Gamma(X \times S)$ is lower semi-continuous. Now $LG(\Delta(x,s)) = \Delta(x,s)$ where $L$ is as in 1.15. Since $L$ is continuous, $\Delta$ is continuous. The rest is clear. []

2.3. PROPOSITION. Let $X$ be a compact space, $S, T \in \underline{CL}$. For each hemimorphism $F: X \times S \longrightarrow T$ and each $f \in LC(X,S)$ we write $\phi(f) = \phi_F(f) = \inf_{x \in X} F(x,f(x)) \in T$. Then

(i) $\phi: LC(X,S) \longrightarrow T$ is a $\underline{CL}$-morphism,

(ii) the diagram

$$\begin{tikzcd} X \times S \arrow[r, "\Delta"] \arrow[dr, "F"'] & LC(X,S) \arrow[d, "\phi_F"] \\ & T \end{tikzcd}$$

commutes,

(iii) $\phi$ is the only $\underline{CL}$-morphism making the diagram in (ii) commutative.

In other words, there is a canonical bijection $F \longmapsto \phi_F$: $\mathrm{Hem}(X \times S, T) = \underline{CL}(LC(X,S),T)$.

Proof. First we prove (ii): $F(y, \Delta(s,x)(y)) = F(x,s)$ if $y = x$

and = 1 if $y \not\leq x$ . Thus $\phi(\Delta(x,s) = F(x,s)$. Assertion (iii) is clear from the fact that $\{\Delta(x,s) : (x,s) \in X \times S\}$ is an order generating set of $LC(X,S)$ (and in particular a generating set). Remains to show (i): We calculate the left adjoint $d: T \longrightarrow LC(X,S)$ of $\phi$. Let $f \in LC(X,S)$ , $t \in T$ . Then $\phi(f) \geq t$ iff $\inf_{x \in X} F(x,f(x)) \geq t$ iff $F(x,f(x)) \geq t$ for all $x$, iff $f(x) \geq \inf\{s \in S : F(x,s) \geq t\}$ (for all $x \in X$) [ since $s \longmapsto F(x,s)$ is in CL]. So we define $d(t)(x) = \inf\{ s \in S : F(x,s) \geq t \}$. Since $G(d(t)) = \{(x,s) : d(t)(x) \leq s\} = F^{-1}(\uparrow t)$ and since F is lower semi-continuous ,$G(d(t))$ is closed,whence $d(t) \in LC(X,S)$ by 1.1.(III).Since $\phi(f) \geq t$ iff $f \geq d(t)$, d is the left adjoint of $\phi$ . By [2, ATLAS] it suffices to show now that $t \ll t'$ implies $d(t) \ll d(t')$ ,which,according to 1.7 is equivalent to $G(d(t') \subseteq \operatorname{int} G(d(t))$. i.e. to $F^{-1}(\uparrow t') \subseteq F^{-1}(\uparrow t)$. But this follows from the upper semi-continuity of F in view of $t \ll t'$ iff $t' \in \operatorname{int} \uparrow t$,i.e. $\uparrow t' \subseteq \operatorname{int} \uparrow t$ (See 1.1.(II)). □

2.4. LEMMA. Let J be a set and $S \in$ CL. Suppose that $\{f_j : j \in J\}$ is a family of morphisms $f_j : S \longrightarrow T$. Then there is a unique continuous hemimorphism $F: J \times S \longrightarrow T$ such that the diagram

$$\begin{tikzcd} J \times S \arrow[r] \arrow[dr, "(j,s)\longmapsto f_j(s)"'] & \beta J \times S \arrow[d, "F"] \\ & T \end{tikzcd} \qquad (D)$$

commutes.

Proof. The existence of a continuous function F making (D)commutative is immediate from the fact that for a compact S the space $\beta J \times S$ is canonically homeomorphic to $\beta(J \times S)$. Let $x \in \beta J$, then there is a net $j_\alpha \in J$ converging to x (where we identify J with a subset of $\beta J$ in the obvious fashion). If $s,t \in S$ then $F(x,s)F(x,t) = \lim F(j_\alpha,s) \lim F(j_\alpha,t) = \lim f_{j_\alpha}(s) f_{j_\alpha}(t) = \lim f_{j_\alpha}(st) = F(x,st)$. □

Now we are ready to calculate arbitrary copowers of an arbitrary <u>CL</u>-object S.

2.5 . **THEOREM** . Let J be a set and $S \in \underline{CL}$. Then the copower ${}^{J}S$ in <u>CL</u> is canonically isomorphic to $\underset{S}{\mathrm{LC}(\beta J,S)}$ , and the j-th coprojection is given by $s \longmapsto \Delta(j,s): X \longrightarrow \mathrm{LC}(X,S)$.

Specifically, let $\{\varphi_j : j \in J\}$ be a family of morphisms $\varphi_j : S \longrightarrow T$ in <u>CL</u>. Then there is a unique morphism $\phi : \mathrm{LC}(X,S) \longrightarrow T$ such that $\varphi_j(s) = \phi(\Delta(j,s))$ for all $j \in J$ and $s \in S$; moreover $\phi$ is given by $\phi(f) = \inf_{j \in J} \varphi_j(f(j))$ .

Proof. By 2.4 we obtain a unique hemimorphism $F: J \times S \longrightarrow T$ extending the function $(j,s) \longmapsto \varphi_j(s)$. By 2.3 there is a unique morphism $\phi = \phi_F : \mathrm{LC}(X,S) \longrightarrow T$ with $F = \phi\Delta$ . Thus $\phi$ is a unique morphism satisfying $\varphi_j(s) = \phi(\Delta(j,s))$ for all $(j,s) \in J \times S$. By 2.3 we have $\phi(f) = \inf_{x \in \beta J} F(x, f(x))$. Since J is dense in $\beta J$, we conclude $\phi(f) = \inf_{j \in J} F(j, f(j)) = \inf_{j \in J} \varphi_j(f(j))$, since f and hence $x \longmapsto F(x, f(x))$ is lower semicontinuous. $\square$

We should remember that knowing co-powers give us a pretty good hold on co-products in general. If J is a set ,then $\{S_j : j \in J\} \longrightarrow \coprod_J S_j : \underline{CL}^J \longrightarrow \underline{CL}$ is afunctor. The retraction diagram

$$\begin{tikzcd} S_k \arrow[r, "can_k"] \arrow[dr, "1_{S_k}"'] & \prod_J S_j \arrow[d, "pr_k"] \\ & S_k \end{tikzcd}$$

then induces a retraction diagram

$$\begin{tikzcd} \coprod_J S_k \arrow[r, tail] \arrow[dr, "1"'] & {}^J(\prod_J S_j) \arrow[d] \\ & \coprod_J S_k \end{tikzcd}$$

which pretty much reduces the question of coproducts to products and copowers. In fact in such categories as <u>CL</u> the co-product $\coprod_J S_k$ is the identified with that subobject of ${}^J(\prod_J S_j)$ which is generated by the images of $S_k \xrightarrow{can_k} \prod_J S_j \xrightarrow{copr_k} {}^J(\prod_J S_j)$. At this point we do not elaborate further what this means in 2.5.