SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

| NAME(S) KEIMEL - MISLOVE | DATE M | D | Y |
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| | September | 3o | 76 |

TOPIC Several Remarks

REFERENCE


[1] SCS-Memoir GIERZ, HOFMANN, KEIMEL, MISLOVE Relations with the interpolation property 8/1/76

[2] SCS-Memoir HOFMANN The space of lower semicontinuoous functions into a CL-object 9/2o/76

[3] HOFMANN, STRALKY ATLAS, Diss. Math. 137 (1976), 1-54

[4] GIERZ, KEIMEL A lemma on primes and its appearance in algebra and analysis, THD Preprint 76

[5] HOFMANN, MISLOVE Commentary on Scott's function spaces SCS-Memoir from July 7, 76.


CONTENTS:

1. The closed subsemilattices of a continuous lattice form a continuous lattice
2. When do the prime elements of a distributive lattice form a closed subset
3. Remarks on lower semicontinuous function spaces
4. Remarks on the continuity of the congruence lattice of the continuous lattice L

These notes report on some discussions during the stay of Mike Mislove at Darmstadt in september 76 . Thanks to A. von Humboldt.


West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

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LSU Baton Rouge (Lawson)
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## 1. <u>The closed subsemilattices of a continuous lattice form a continuous lattice</u>.

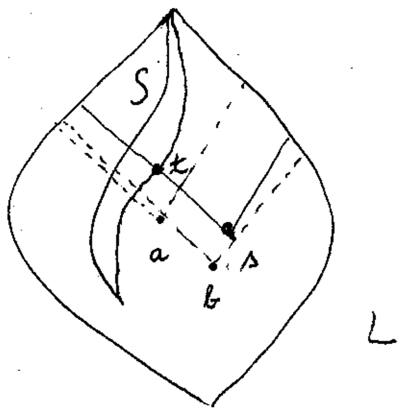
Let $L$ be a continuous lattice endowed with the Lawson topology. Let $Sub(L)$ denote the set of all closed subsemilattices of $L$ ordered opposite to inclusion. The closed subsets of $L$ form a continous lattice and we have the operator that associates the generated closed subsemilattice with every closed subset of $L$. By lemma 1.13 in [1]

$Sub(L)$ will be a continuous lattice

if we can show that every closed subsemilattice $S$ of $L$ has a neighborhood basis of closed subsemilattices. This might be known, but I do not remember having seen it. By compactness it will suffice to prove the following:

Claim: Let $S$ be a closed subsemilattice of $L$ and $s$ an element of $L$ <u>not</u> in $S$. Then there is a closed subsemilattice $T$ which is a neighborhood of $S$ and does not contain $s$.

Proof. Let $t = \inf\{x \in S \mid s \leqslant x\}$. Choose an element $a \ll t$ with $a \nleq s$. The compact set $S \setminus \Uparrow a$ is covered by the updirected union of the sets $L \setminus \uparrow b$ with $b \ll s$. Thus there is a $b \ll s$ such that $L \setminus \uparrow b$ contains $S \setminus \Uparrow a$. Let $T$ be the union of the closed filter $\uparrow a$ and the closure of the order ideal $L \setminus \uparrow b$. Then $T$ has the desired property.



## 2. When do the prime elements of a distributive continuous lattice form a closed subset.

In section 3 of [2] it was proved that a continuous distributive lattice $L$ is isomorphic to the lattice of all open decreasing subsets of its set $P(L)$ of prime elements, provided that $P(L)$ is closed in $L$. Here and in the sequel, $L$ will be endowed with its Lawson topology.

Here we want to give an internal criterion for the set $P(L)$ of prime elements of a ditributive continuous lattice to be closed.

2.1. PROPOSITION. In a distributive continuous lattice $L$ the set $P(L)$ of prime elements is closed, if and only if the following condition (C) is satisfied:

(C) For all $a,c,d$ in $L$, $a \ll c$ and $a \ll d$ imply $a \ll c \wedge d$.

Proof. If $P(L)$ is closed, then $L$ is isomorphic to the lattice $D(P(L))$ of all decreasing open subsets of $P(L)$ by [2], Theorem 3.9. In this lattice one has $U \ll V$ iff $\overline{U}^d \subseteq V$, where $\overline{U}^d$ denotes the decreasing subset of $P(L)$ generated by the closure of $U$. Clearly, if $\overline{U}^d \subseteq V$ and $\overline{U}^d \subseteq W$, then $\overline{U}^d \subseteq V \cap W$. Thus (C) is satisfied.

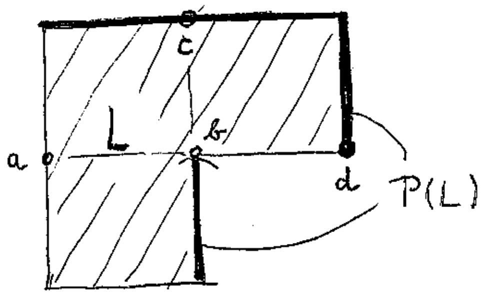
For the converse, we first note that the set $P(J(L))$ of all prime ideals of $L$ is closed in the algebraic lattice $J(L)$ of all (lattice) ideals of $L$. Indeed, $P(J(L))$ with the induced Lawson topology is homeomorphic to the space of all lattice homomorphisms $f: L \longrightarrow \mathbf{2}$ endowed with the topology of pointwise convergence, and this latter space is compact. The map $I \longmapsto \sup I$ from $J(L)$ onto $L$ is continuous (see [3]). Consequently, the image of $P(J(L))$ is closed. Thus, $P(L)$ will be closed, if we can show that $\sup I$ is a prime element of $L$ whenever $I$ is a prime ideal.

So, let $I$ be a prime ideal of $L$ and $p = \sup I$. Let $a,b$ be elements of $L$ such that $a \wedge b \leq p$. Suppose that $a \nleq p$ and $b \nleq p$. Then there are elements $c \ll a$ and $d \ll b$ such that $c \nleq p$ and $d \nleq p$. From condition (C) we conclude that $c \wedge d \ll a \wedge b$. AS $a \wedge b \leq p = \sup I$, we conclude that $c \wedge d$ belongs to the ideal $I$. As $I$ is prime, $c \in I$ or $d \in I$, whence $c \leq p$ or $d \leq p$, a contradiction.

2.2. COROLLARY. In a distributive algebraic lattice the set of prime ~~ideals~~ elements is closed for the Lawson topology, if and only if the meet of any two compact elements is compact.

2.3. EXAMPLE. Let $L$ be the following subset of the square $[0,1]^2$ ordered coordinatewise:

$$L = ([0,\tfrac{1}{2}]\times[0,1]) \cup ([\tfrac{1}{2},1]\times[\tfrac{1}{2},1])$$




Then $L$ is a distributive continuous lattice. The set of prime elements $P(L) = ([0,1]\times\{1\}) \cup (\{1\}\times[\tfrac{1}{2},1]) \cup (\{\tfrac{1}{2}\}\times[0,\tfrac{1}{2}[)$ is not closed. In fact, the point $b = (\tfrac{1}{2},\tfrac{1}{2})$ is not prime, although a limit of primes. In order to se that condition (C) fails, consider $a = (0,\tfrac{1}{2})$ , $c = (\tfrac{1}{2},1)$ , $d = (1,\tfrac{1}{2})$ . Then $a \ll c$ and $a \ll d$ . But $a \ll c \wedge d = b$ does <u>not</u> hold.

2.4. One can show more generally that in a distributive continuous lattice $L$ the closure of the set $P(L)$ of all prime elements is exactly the set of all elements which are suprema of prime ideals of $L$ .

2.5. QUESTION. What does condition (C) mean for arbitrary continuous lattices.

## 3. Remarks on lower semicontinuous function spaces.

Let $X$ be a compact Hausdorff space and $L$ a continuous lattice. We use the terminology and notation of [2]. We propose a proof of Theorem 1.1o in [2] which is based on proving first 1.15 of [2]. $L$ is endowed with its Lawson topology.

2.1. DEFINITION. An upper graph is a subset $G \subseteq X \times L$ which is closed and which has the property that for all $x$ in $X$ the set

$$G_x = \{a \in L \ ; \ (x,a) \in G\}$$

is a filter of $L$ and nonempty.

2.2. NOTE: If $f:X \longrightarrow L$ is lower semicontinuous, the set

$$G(f) = \{(x,a) \ ; \ f(x) \leq a\}$$

is an upper Graph in the sense of 2.1. Conversely, if $G$ is an upper graph, the function $f:X \longrightarrow L$ defined by

$$f(x) = \inf G_x \quad \text{for all } x \text{ in } X$$

is lower semicontinuous. We conclude that the lattice $LC(X,L)$ is isomorphic to the lattice $G(X,L)$ of all upper graphs ordered by the opposite of inclusion.

2.3. NOTE: The intersection of any family of upper graphs is still an upper graph. Consequently, on the continuous lattice $\Gamma(X \times L)$ of all closed subsets of $X \times L$ (ordered by the opposite of inclusion) (we have a kernel operator) which associates with each closed subset the smallest upper graph containing it. By lemma (1.13) in [1], the following will immediately imply the non-trivial part of Theorem 1.1o and 1.15 in [2]:

2.4. LEMMA. Every upper graph has a neighborhood basis of upper graphs.

Proof. By compactness, it suffices to show the following: Let $G$ be an upper graph. Let $(x,a) \in X \times L \setminus G$. Then there is an upper graph $H$ which is a neighborhood of $G$ but does not contain $(x,a)$. If $(x,a)$ does not belong to $G$, then $\inf G_x \not\leq a$. Choose an element $b$ in $L$ with $b \ll \inf G_x$ but $b \not\leq a$. By condition (3') of lemma 1.1 in [2] there is a neighborhood $V$ of $x$ such that $b \ll f(u)$ for all $u$ in $U$. Let $W$ be a neighborhood of $x$ the closure of which is still in $U$. Then $H = (W \times \uparrow b) \cup ((X \setminus W) \times L)$ is an upper graph with the desired properties.

## 4. Congruences on a CL-object

I. Let $S \in \underline{CL}$, and let Cong(S) denote the family of all closed congruences on S. We first determine sufficient conditions on S under which (Cong(S), $\vee$) is a $\underline{CL}$-object, where $\rho \vee \theta$ is the closed congruence generated by the congruences $\rho$ and $\theta$. Since $\vee$ is our principal operation, it follows that $\rho \leq \theta$ iff $\theta \subseteq \rho$.

Lemma 1. For $\rho, \theta \in$ Cong(S), we have $\rho \ll \theta$ if $\theta \subseteq \rho^\circ$.
Proof. If $\theta \subseteq \rho^\circ$, then given $D \uparrow \subseteq$ Cong(S) with $\theta \leq \sup D$, we note that $\sup D = \bigcap D$, and so $\rho$ is a neighborhood of $\bigcap D$. As each $\delta \in D$ is a compact subset of $S \times S$, it then follows that there is some $\delta \in D$ with $\delta \subseteq \rho$; i.e., $\rho \leq \delta$. □

Now, if $S \in \underline{Z}$, then $S = \lim S/\rho_i$, where $S/\rho_i$ is finite for each i. Thus $\rho_i$ is a neighborhood of $\Delta(S)$, the diagonal of $S \times S$, so that $\rho_i \ll \Delta(S)$ for each i, by the Lemma. Moreover, $S = \lim S/\rho_i$ implies that $\Delta(S) = \bigcap \rho_i$, and so $\Delta(S) = \sup \Downarrow\Delta(S)$. Finally, if $S/\rho \in \underline{Z}$ for some congruence $\rho$ on S, the same argument--this time applied to $S/\rho$--shows that $\rho = \sup \Downarrow \rho$. It is then clear that for a stable $\underline{Z}$-object S(i.e., one for which all quotients are also $\underline{Z}$-objects), we have (Cong(S), $\vee$) $\in \underline{CL}$.

Note also that, for an $S \in \underline{CL}$ which is not a stable $\underline{Z}$-object, there is some congruence $\rho$ on S with $S/\rho \equiv I$, the unit interval. Now, if (Cong(S), $\vee$) $\in \underline{CL}$, then (Cong(I), $\vee$) $\equiv \uparrow \rho$ is also a $\underline{CL}$-object. [But, the only closed congruence on I which is also a neighborhood of the diagonal is the universal congruence, and it then follows that for some $\theta \in$ Cong(S) with $\theta \ll \rho$, it is not true that $\rho \subseteq \theta^\circ$. Thus, if S is a $\underline{CL}$ object which is not a stable $\underline{Z}$-object, and if (Cong(S), $\vee$) $\in \underline{CL}$, then there are congruences $\rho$ and $\theta$ on S with $\theta \ll \rho$ but $\rho \nsubseteq \theta^\circ$. As is shown in [5], it turns out that (Cong(S), $\vee$) $\in \underline{CL}$ iff S is a stable $\underline{Z}$-object, and, in this case, Cong(S) is itself a stable $\underline{Z}$-object. However, we do not see how to obtain this result in full from these methods. ] see ADDENDUM p. 7

II. We now turn our attention to the other obvious operation on Cong(S): intersection. From an algebraic point of view, this is usually thought of as the principal operation on the congruence lattice; however, since intersection is never a continuous operation on the closed subsets of a topological space--well, almost never, and since union is a continuous operation, intersection is not the most appealing operation from the topological viewpoint. However, our results here are not much better than those obtained for the case of union given above.

Again, let $S \in \underline{CL}$, and let Cong(S) denote the lattice of all closed congruences on S. Then, for congruences $\rho$ and $\theta$ on S, since $\cap$ is our principal operation this time, we have $\rho \leq \theta$ iff $\rho \subseteq \theta$ . We first consider the case of the unit interval I.

Lemma 2. Let $\omega$ be the universal congruence on I, and suppose that $\theta$ is another congruence on I which is not the diagonal on I. Then $\theta \not\ll \omega$ .

Proof. Recall that, for any congruence $\rho$ on I, the congruence classes decompose I into a disjoint family of subintervals. Since $\theta$ is not the identity congruence, there are $a,b \in I$ with $a < b$ and $[a,b]$ a $\theta$-class. Moreover, since $\theta$ is not the universal congruence on I, either $0 < a$ or $b < 1$. Choose sequences $a_n < b_n$ with $a < a_n < b_n < b$ and $a = \lim a_n$ and $b = \lim b_n$; define a sequence of closed congruences $\theta_n$ on I by $\theta_n = ([0,a] \times [0,a]) \cup ([a_n,b_n] \times [a_n,b_n]) \cup ([b,1] \times [b,1]) \cup \Delta(S)$. Then clearly $\theta \not\subseteq \theta_n$ for each n, but one readily sees that $\sup \theta_n$ is the universal congruence, since $(0,a)$, $(a,b)$, and $(b,1)$ must be in $\sup \theta_n$. □

Now, if $S \in \underline{CL}$, and $(\mathrm{Cong}(S), \cap) \in \underline{CL}$, then for any congruence $\rho$ on S, it is clear that $(\mathrm{Cong}(S/\rho), \cap) \cong \uparrow\rho$, and so $(\mathrm{Cong}(S/\rho), \cap)$ is also a $\underline{CL}$-object. Since any $\underline{CL}$-object which is not a stable $\underline{Z}$-object admits the unit interval as a quotient, the Lemma then implies that any $S \in \underline{CL}$ with $(\mathrm{Cong}(S), \cap)$ also in $\underline{CL}$ must itself be a stable $\underline{Z}$-object.

There are some examples of $S \in \underline{CL}$ for which $(\mathrm{Cong}(S), \cap) \in \underline{CL}$: Trivially finite lattices will do. But also the one point compactification of $\mathbb{N}$ has this property. One may conjecture that all examples have to be almost discrete as the preceeding example.

ADENDUM to part I. In order to show that a non-stable $\underline{Z}$-object has its congruence lattice not in $\underline{CL}$ , it suffices to show that $(\mathrm{Cong}(I), \vee) \notin \underline{CL}$ . For this let $\rho$ be any congruence (closed) on I different from the all-congruence. It is easily seen, that there is a closed intervall $[a,b]$ with $a \neq b$ such that all $\rho$-classes of elements in $[a,b]$ are singleton. Let $a_n$ be a sequence decreasing to a and $b_n$ a sequence increasing to b with $a_1 = b_1$ . Let $\rho_n$ be the congruence which has the interval $[a_n,b_n]$ as the only non-singleton class. Then the $\rho_n$ form a sequence decreasing to the diagonal, but no $\rho_n$ contains $\rho$ . Hence, $\rho \not\ll \Delta$ for all $\rho$ different from the universal congruence.