MEMORANDUM

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Lieber Herr Keimel:
Ihre Seminarnotizen über das Jonsson Lemma sind sehr interessant. Im augenblick habe ich dazu nichts hinzuzufügen. Anbei Kommentare zu einem Bündel Notizen von Jimmie D.Lawson, die in meiner Perspektive sehr nahe an GK herankommen.
Beachten Sie,dass ich einige Beobachtungen aus NOTES GK (meine Notizen zu Ihrer Arbeit) nun auf CL ausgedehnt und verschärft habe.

Achten Sie insesondere auf die Beziehungen "order generating" - "generating" usw.
Die Frage,die ich in NOTES GK über koprodukte stellte (Seite 10,letzter Absatz) ist durch ein Beipliel negativ beantwortet.
Was ich jetzt hauptsächlich wissen möchte, ist ob jeder CL-Halbverband eine eindeutige kleinste abgeschlossene Erzeugendenmenge besitzt.

Herzliche Grüsse
Ihr

Karl Heinrich H.

NOTES ON NOTES BY JDL (concerning what he calls the 'spread') by K.H. Hofmann

Sources of reference as usual:
HMS DUALTIY, HS ATLAS,
but also my recent notes on Gierz and Keimel 'Topologische Darstellung', which I will abbreviate NOTES GK.
Reference to Jimmie's notes is by JDL.

There appears to be a non-trivial ~~xxxxip~~ overlap between JDL and GK although the objectives appear to be different.
I recall a few things from NOTES GK:

A set $X \subseteq S$ (in a semilattice S) is <u>order generating</u> iff

$$s = \inf(\uparrow s \cap X) \quad \text{for all } s \in S.$$

If S is a topological semilattice we say that X is generating, if S is the smallest closed subsemilattice containing X.

ORDER GENERATING IS STRONGER THAN GENERATING.

Section 1. IRR

The set of all <u>completely</u> meet irreducible elements in a semilattice L will be denoted Irr L, the set of all meet irreducibles will be called IRR L, and the set of all primes is PRIME L. (I guess if they played a role, I would denote the set of complete primes by Prime L.) The closure of Irr T in a topological semilattice T will be written $\overline{\text{Irr}}$ T etc. We observed in NOTES GK

PROPOSITION 1. Let $T \in \underline{Z}$ and $X \subseteq T$. Then these are equivalent:

(i) X is order generating.

(ii) Irr $T \subseteq X$.

(iii) $\uparrow s \cap X = \uparrow t \cap X \Rightarrow s = t$ for all $s, t \in T$.

The following are aslo equivalent and ~~ixpixxkx~~ follow from the preceding:

(1) X is generating

(2) $\uparrow h \cap X = \uparrow k \cap X \Rightarrow h = k$ for all $h, k \in K(T)$

(3) $k = \inf(\uparrow k \cap X)$ for all $k \in K(T)$.

If, in addition, T is also distributive, then (1),(2),(3) are also equivalent to

(4) $\overline{X}$ is order generating (i.e. Irr $T \subseteq \overline{X}$)

(5) PRIME $(T) \subseteq \overline{X}$.

In particular, in the last case, $\overline{\text{Irr}}\ T = \overline{\text{PRIME}}\ T$ is the unique smallest closed generating set and unique smallest closed order generating set.

For these matters see NOTES GK 2.1 through 2.4. I note, however, that the relations (i)<=>(iii) =>(ii) hold in $\underline{CL}$.
In order to expand the theory from $\underline{Z}$ to $\underline{CL}$ (as everybody does these days) it is clear that Irr is no longer sufficient, as the example Irr I = {1} shows. I therefore want to develop some remarks on IRR .

REMARK 2. Let $S \in \underline{S}$, $x \in S$.

a) $x \in$ IRR S iff $\uparrow x \setminus \{x\}$ is a semigroup iff $\uparrow x \setminus \{x\}$ is a filter. *iff* $x \in$ PRIME $\uparrow x$.

b) If U is a filter of S and x is maximal in $S \setminus U$, then $x \in$ IRR S.

Proof. a) is immediate from the definition.

b) If x is maximal in $S\setminus U$, then $\uparrow x \setminus \{x\} \subseteq U$, thus $\uparrow x \setminus \{x\} = U \cap \uparrow x$ is a filter, and the assertion follows from a.

LEMMA 3. Let T be a compact semilattice, $t \in T$ and U an open filter with $t \notin U$. Then there is an $x \in$ IRR T with $t \leq x$ and $x \notin U$.

Proof. The set $\uparrow t \cap (T\setminus U)$ is a compact poset, hence has a maximal element x. By 2.b, $x \in$ IRR T.

PROPOSITION 4. For $T \in \underline{CL}$, IRR T is a generating set, i.e.

$t = \inf(\uparrow t \cap \text{IRR } T)$ for all $t \in T$.

Proof. Let $t \in T$ and set $s = \inf \uparrow t \cap \text{IRR } T$. (Recall $\inf \phi = 1$!). Clearly $t \leq s$. Assume $t < s$. Since $T \in \underline{CL}$ there is an open filter U with $t \notin U$ and $s \in U$. By Lemma 3, there is an $x \in \uparrow t \cap \text{IRR } T$ with $x \notin U$. Then $s \leq x$ by definition of s, whence $x \in U$ since $s \in U$ and U is a filter. Contrariction. □

LEMMA 5. Let X be an order generating set in a $\underline{CL}$-object T. Then

PRIME T $\subseteq \overline{X}$.

Proof. NOTES GK 2.3: By "THE LEMMA"(as it is now called by GK), if $p \in$ PRIME T and $p = \inf(\uparrow p \cap X)$ then $p \in \overline{X}$. □

PROPOSITION 6. Let $T \in \underline{CL}$, $X \subseteq T$. Then (n)=>(n+1):

(1) IRR T $\subseteq$ X , (2) X is order generating

~~(3)xxxxxxxxxxxxxx~~

(3) $\overline{X}$ is order generating, (4) PRIME $T \subseteq \overline{X}$.

Now suppose that T is distributive. Then (3),(4) and (5) below are equivalent

(5) IRR $T \subseteq \overline{X}$.

Proof. Put thre preceding results together with the fact that for distributive lattices IRR = PRIME. □

PROPOSITION 6a. Let $T \in \underline{CL}$. Then these statements are equivalent: (1) IRR T = PRIME T, (2) T is distributive (3) T is Brouwerian.
Proof. (2)⇔(3) is known (see e.g. HK DUALITY p. 55). (2) ⇒ (1) standard.
(1) ⇒ (2): By Prop 6 and (1), PRIME T is order generating. This is known to imply (2).

~~Thix~~ We turn to the question of generation.

DEFINITION 7. Set $A(T) = \{a \in T: a = \inf \operatorname{int} \uparrow a\}$. (ATLAS).

If $T \in \underline{CL}$, then $t = \sup(\downarrow t \cap A(T))$ for all $t \in T$ since $\inf U \in A(T)$ for every open filter U of T.

LEMMA 8. Let $T \in \underline{CL}$ and $X \subseteq T$. Then the following are equivalent: (Y^{o} = int Y):

(1) $a = \inf((\uparrow a)^{o} \cap X)$ for all ~~a~~ $a \in A(T)$.

($\overline{1}$) ~~(2)~~ $a = \inf((\uparrow a)^{o} \cap \overline{X})$ for all $a \in A(T)$.

(2) $(\uparrow a)^{o} \cap X = (\uparrow b)^{o} \cap X \Rightarrow a = b$ for all $a, b \in A(T)$.

($\overline{2}$) same as (2) but with $\overline{X}$ replacing X.

(3) X is generating.

($\overline{3}$) $\overline{X}$ is generating.

Proof. ~~xx~~ (1)=>($\overline{1}$) trivial.

~~($\overline{1}$)=>(1): By ($\overline{1}$) , a is approximated from above by elements $\bigwedge F$ where F is a finite set in $(\uparrow a)^{o} \cap \overline{X}$; since $(\uparrow a)^{o}$ is open it follows that a is approximated from above by elements $\bigwedge F$ where F is finite in $(\uparrow a)^{o} \cap X$.~~

(3) ⇔ ($\overline{3}$) is trivial.

(1) => (2) trivial;

(not 1) => (not (2)): If(not 1) there is an $a \in A(T)$ with $a < b = \inf((\uparrow a)^{o} \cap X)$ .But $a \leq b$ implies $(\uparrow b)^{o} \cap X \subseteq (\uparrow a)^{o} \cap X$ and the definition of b implies ~~xxx~~ $(\uparrow a)^{o} \cap X \subseteq (\uparrow b)^{o} \cap X$.

We have proved (not 2).

(1) =>(3) is trivial since A(T) is dense in T.

(3) => (1) . Let $a \in A(T)$. Let U be a neighborhood of a.

Then $U \cap (\uparrow a)^{o} \neq \emptyset$ by the definition of A(T). By (3) there is a finite set $F \subseteq X$ with $\bigwedge F \in U \cap (\uparrow a)^{o}$. Thus (1) follows □.

LEMMA 9. Let $T \in \underline{CP}$, and let $X = \overline{X} \subseteq T$. Define $\underline{X} = \{t \in T : t = \inf \Uparrow t \cap X\}$ Then $\underline{X}$ is closed.

Proof. Let $s \in (\underline{X})^-$. For each entourage U of the uniform structure of T we pick a $t_U \in \underline{X}$ with $t_U \in U(s)$. Then there is a finite set $F_U \subseteq X$ with $F_U \subseteq \uparrow t_U$ and $(t_U, \bigwedge F_U) \in U$. By compactness, there is a cofinal function $j \mapsto U(j)$ on a directed set J with values in the uniform structure of T such that $G_j = F_{U(j)}$ converges to a closed subset G in the compact space X relative to the Hausdorff topology on X. Each $g \in G$ is the limit of a net $g_j \in G_j$. From $t_{U(j)} \leq g_j$ we conclude $s = \lim t_{U(j)} \leq \lim g_j = g$, i.e. $G \subseteq \uparrow s \cap X$. For any CP-object T, the function $A \mapsto \bigwedge A : c(T) \to T$ is continuous (in fact this is characteristic for $\underline{CP}$). Hence $\bigwedge G = \lim \bigwedge G_j$. But $(t_{U(j)}, \bigwedge G_j) \in U(j)$, whence $(s, \bigwedge G) = \lim(t_{U(j)}, \bigwedge G_j) \in U(j)^-$ for all j. Since $j \mapsto U(j)$ is cofinal, we have $(s, \bigwedge G) \in \cap\{U(j)^- : j \in J\} = \mathrm{diag}_{T \times T}$, i.e. $s = \bigwedge G$. Since $G \subseteq \uparrow s \cap X$ we conclude $s = \inf(\uparrow s \cap X)$. □

THEOREM ~~PROPOSITION~~ 10. Let $T \in \underline{CP}$ and $X \subseteq T$; X is generating, iff $\overline{X}$ is order generating. In particular, a closed set is generating iff it is order generating.

Proof. (i) Let X be generating. By LEMMA 9 above $(\underline{\overline{X}})$ is a closed subset which contains $A(T)$ by LEMMA 8. Since $A(T)$ is dense, then $T \subseteq (\underline{\overline{X}})$, which by the definition of $(\underline{\ \ })$ means that $\overline{X}$ is order generating in T.

ii) If $\overline{X}$ is order generating, then $\overline{X}$ is generating (Lemma 8), whence X is generating. □

COROLLARY 11. If X is generating in $T \in \underline{CL}$, then $\mathrm{Irr}\, T \cup \mathrm{PRIME}\, T \subseteq \overline{X}$.

Proof. Theorem 10 and Lemma 5 (plus (i)⇒(ii) in Prop. 1, which holds always. □

PROPOSITION 12. Every $\underline{CP}$ object has minimal closed order generating sets.

Proof. (Indication.) Let X_j be a tower of closed order generating sets. Use the method of proof of LEMMA 9 to show that $\cap X_j$ is still order generating. □

~~finite set $F \subseteq X$ with $\uparrow \bigwedge F \subseteq U \cap (\Uparrow a)^{\circ}$. Thus (1) follows.~~ $\square$

PROPOSITION 13. Let $T \in \underline{CL}$ be distributive. Then the following statements are equivalent for $X \subseteq T$:

(1) X is generating.

(2) $\overline{X}$ is order generating. ~~(xxxx~~

(3) $\mathrm{IRR}\ T = \mathrm{PRIME}\ T \subseteq \overline{X}$.

Proof. By Prop. 6 we have (2)<=>(3) and by THEOREM 10, (2)<=>(1) ~~[illegible]~~.

~~Suppose (1) and let $p \in \mathrm{PRIME}\ T$. Take an arbitrary $a \in A(T)$ with $a \ll p$. By Lemma 8 we have $a = \inf (\Uparrow a)^{\circ} \cap X$. By THE LEMMA there is an $x_a \in ((\Uparrow a)^{\circ} \cap X)^{-} \subseteq \uparrow a \cap \overline{X}$ with $x_a \leq p$. Since p is approximated from below with $a \ll x$, then $p \in \overline{X}$. Thus (3) follows.~~ $\square$

ZUSATZ 14. If $T \in \underline{Z}$ is distributive and $X \subseteq T$, (1),(2),(3) of 9 are also equivalent to

(4) $\mathrm{Irr}\ T \subseteq \overline{X}$.

Proof. Prop. 1. $\square$

COROLLARY 15 . In any distributive $T \in \underline{CL}$ the set $\overline{\mathrm{IRR}\ T} = \overline{\mathrm{PRIME}\ T}$ is the unique smallest closed (order) generating set ~~[illegible]~~ ~~[illegible]~~ If ~~even~~ $T \in \underline{Z}$ then $\overline{IRR(T)} =$ ~~[illegible]~~ $\overline{\mathrm{Irr}\ T}$, is the unique smallest closed generating set. If $T \in \underline{Z}$ is distributive, then $\overline{IRR(T)} = \overline{Irr(T)} = \overline{PRIME(T)}$. $\square$

Does anyone know whether the relation $\overline{\mathrm{IRR}} = \overline{\mathrm{Irr}}$ holds in $\underline{Z}$ even without distributivity?

This assertion is equivalent to the following:

If $0 \in \overline{\mathrm{PRIME}\ T}$, $T \in \underline{Z}$, then $0 \in \overline{\mathrm{Irr}\ T}$.

Any proofs? Counterexamples?

Let T be a compact semilattice.
Consider

(I) $T \in CL$;

(II) $IRR\,T$ is order generating.

We have seen (I) $\Rightarrow$ (II). How about the converse?

Is there a $T \in CL$ with more than one minimal closed order generating set? (By 15, such an example cannot be in $\underline{Z}$ or distributive

Section 2

Variations of a theme by GK + JDL

Let $T \in \underline{CL}$. Let Sub(T) the compact semilattice of all closed subsemilattices of T under the multiplication $(A,B) \longmapsto AB$. Notice that $A \leq B$ in Sub(T) iff $B \subseteq A$. Wenote $\bigvee \{A_j : j \in J\} = \cap_J A_j$ in Sub(T). As a consequence, $A \ll B$ iff for every family $C_j, j \in J$ in Sub(T) with $\cap_j C_j \subseteq B$ thereis a finite set $F \subseteq J$ with $\cap_F C_j \subseteq A$. This is satisfied if $B \subseteq \operatorname{int} A$; but since $T \in \underline{CL}$, then B has a basis of semilattice neighborhoods, and in taking $\{C_j : j \in J\}$ to be the family of all compact semilattice neighborhoods of B we see that this condition is also necessary. Thus $A \ll B$ iff $B \subseteq \operatorname{int} A$.
By what we just observed (namely, that B has a basis of semilattice neighborhoods) we know that $B = \sup \{A : A \ll B\}$.
According to ATLAS , this makes Sub(T) a $\underline{CL}$ -object. We have:

PROPOSITION 1 . Let $T \in \underline{CL}$, then $\mathrm{Sub}(T) \in \underline{CL}$, and $A \ll B$ in Sub(T) iff $B \subseteq \operatorname{int} A$.

Let us note in passing that the function $x \longmapsto \uparrow x : T \longrightarrow \mathrm{Sub}(T)$ is algebraically a morphism iff $\uparrow xy \subseteq (\uparrow x)(\uparrow)$ for all $x,y \in T$ (for the convers in clusion is always true). If T is distributive, then this condition is satisfied:Indeed if $xy \leq z$, then $z = (x \vee z)(y \vee z) \in (\uparrow x)(\uparrow y)$. Thus:

ZUSATZ 2. If T is distributive, then $x \longmapsto \uparrow x : T \longrightarrow \mathrm{Sub}(T)$ is an embedding algebraically. □

PROPOSITION 3. The mapping $\uparrow : T \longrightarrow \mathrm{Sub}(T)$ is a morphism in $\underline{CL}^{op}$, hence preserves arbitrary sups, repsects $\ll$, is continuous from below and lower semicontinuous. It is right adjoint to the map $\min : \mathrm{Sub}(T) \longrightarrow T$ which therefore is a CL=morphism.

Proof. We have $A \geq \uparrow x$ iff $A \subseteq \uparrow x$ iff $\min A \geq x$, whid shows that $\uparrow$ is right adjoint to min (ATLAS). The remainder follows from ATLAS.

EXAMPLE 4. Let $T = \{(x,y) \in I \times I : x=y = 1/n, n=1,2,\ldots,\ x=y=0$ or $x=0, y = 1\}$. Then $(1/n,1/n) \longrightarrow (0,0)$,but $\uparrow(1/n,1/n) \not\longrightarrow \uparrow(0,0)$. □

The example shows that even in the distributive case, $\Uparrow$ does not preserve infs ,hence does not have a right adjoint and is not continuous.

Much of what has been said applies immediately to the semilattice $c(T)$ of all compact subsets of T under multiplication of sets. Clearly, Sub(T) is a subsemilattice of c(T). Note that in fact c(T) also has a CL-operation $\cup$ and $(c(T), \cup, .)$ is a compact semiring.

By ATLAS duality, each $A \in \mathrm{Sub}(T)$ corresponds bijectively to a CL^{op}-congruence on T,namely the kernel congruence of the right adjoint $d_A : T \longrightarrow A$ of the inclusion map $g_A : A \longrightarrow T$. In order to link this observation with NOTES GK ,I note that if $T \in \underline{Z}$ then there is a bijection between the ~~xxxxxxxxx~~ $\underline{CP}^{op}$ -congruences on T and the congruences on $K(T)$ $(\cong \hat{T})$ obtaines simply by restriction $R \longmapsto R \cap (K(T) \times K(T))$ (since $K(d_A): K(T) \longrightarrow K(A)$ is simply ~~xxx~~ $K(d_A) = d_A | K(T)$).

I wish to ~~dwell~~ we for a moment on coproducts in $\underline{CP}$.
Let $\{T_j : j \in J\}$ be a family in CP . We let $T = \coprod_J T_j$ denote ~~the~~ its coproduct and we consider T_i ~~and~~ as embedded into T as the i-th cofactor,i.e. the coprojection $g_j : T_j \longrightarrow T$ is just an inclusion. Let $d_j : T \longrightarrow T_j$ be the right adjoint given by $d_j(t) = \inf(\uparrow t \cap T_j)$ (see (DIMENSIONAL CAPACITY HMS)). Then T is the product of the T_j in CP^{op} with d_j as product projections. Let $X = \cup_J T_j \subseteq T$. Then $X \subseteq \sum_J T_j \subseteq T$, where $\sum_J T_j$ is the algebraic coproduct (in $\underline{S}$) with the colimit topology in the category of topological semilattices (~~x~~perhaps in the category of k-semilattices-I am undecided). Note that CP ~~xxxx~~ has bi-products, i.e. that $A \times B = A \amalg B$ in the obvious fashion. Thus $\sum_J T_j$ is the ascending (up-directed union)of the family of all $\prod_F T_j = \coprod_F T_j$, $F \subseteq J$ finite. Every element of $\sum_J T_j$ is a finite inf of elements in X, in particular, X is order generating in $\sum_J T_j$.

I now want to settle (if I can) the somewhat delicate question whether of not X is order -generating in T. For this purpose I consider the very special case that all T_j have two elements.

The following Lemma helps us to understand coproducts of Z-objects in CL , since we know coproducts in Z reasonably well by HMS DUALITY.

PROPOSITION 5. If $\{T_j : j \in J\}$ is a family of finite objects, then the Z-coproduct S of the T_j and the CP-coproduct T of the T_j agree.

Proof. We must show that S has the universal property of the coproduct in CP . Since I is a co-generator of CL , it suffices to prove the following: For each family $f_j : T_j \longrightarrow I$ of CP-morphisms there is a unique CP -morphism $f : S \longrightarrow T$ such that $f_j = fs_j$,where $s_j : T_j \longrightarrow S$ are the coprojections.

~~or DUALITY HMS IV - 2) [we could also take the more canonical~~

It suffices to produce a Z-object Z and a morphism $g : Z \longrightarrow I$ tgether with a family of morphisms $h_j : T_j \longrightarrow Z$ such that $f_j = gh_j$; for then by the universal property of S there would be a unique $h : S \longrightarrow Z$ with $h_j = hs_j$ for all j, yielding $f = gh$ with $fs_j = ghs_j = gh_j = f_j$, and f would be unique since the $s_j(T_j)$ generate S_j.

Now let $g : Z \longrightarrow I$ e.g. be the Cantor map (DUALITY HMS IV - 2 or DIMENSION RAISING). Then by the finiteness of T_j, the h_j exist as desired.□

[I would like to know whether or not in general Z-coproducts are CP - coproducts. I can see how the above method would still work for a countable family of stable Z -objects.]

This proposition anyhow allows me to treat copowers of $^J2$, i.e. free $\underline{CP}$-objects over $\underline{Set}$. In orther words, if J is a set, then the free $\underline{Z}$-object $F(J) = {}^J2$ is the free $\underline{CP}$-object generated by J with identity $\emptyset$. Now $F(J)$ is $\cong$ $c(\beta J)$ under $\emptyset$, and J is $\cong$ embedded in $F(J)$ via $j \longmapsto \{j\}$. Let $\tilde{J}$ be the image, i.e. write $\tilde{j} = \{j\}$. Let $Q \in c(\beta J)$ be an arbitrary element.

Then $\uparrow Q \cap \tilde{J} = \{P \in c(\beta J) : P \subseteq Q \text{ and } Q \in \tilde{J}\} = \{\{j\}: j \in J$ and $j \in Q\} = (J \cap Q)^{\sim}$ (where we identify J with its image in βJ). Now let $\mathcal{A} \subseteq c(\beta J)$. Then $\inf \mathcal{A} = (\bigcup\{A: A \in \mathcal{A}\})^{-}$. Thus

$\inf(\uparrow Q \cap \tilde{J}) = \inf(J \cap Q)^{\sim} = (J \cap Q)^{-}$ in βJ .

We have shown:

EXAMPLE 6. The set $\bigcup_J T_j$ in the coproduct $\coprod_J T_j$ need not be order generating. In fact let $T_j = 2$ for all j, then $\coprod_J T_j = F(J)$ and $\bigcup_J T_j = J \cup \{1\}$ where J is identified with its image in $F(J)$. In general we have $x < \inf(\uparrow x \cap (J \cup \{1\}))$. Indeed if x is identified with Q under the isomorphism $F(J) \to c(\beta J)$, then equality holds iff $Q \cap J$ is dense in $Q \subseteq \beta J$. If x corresponds to $Q \subseteq \beta J \setminus J$, then $\inf(\uparrow x \cap (J \cup \{1\})) = 1$. □

It is in-structive to observe for an arbitrary compact space E in the $\underline{CL}$-object $F(E) \cong (c(E), \cup)$ the irreducibles: $F(E)$ is distributive, $\operatorname{Irr} F(E) = \operatorname{PRIME} F(E) = E \subseteq F(E)$, This shows, in particular, that Irr is always order generating in $\underline{Z}$-objects, but that this not characeristic for $\underline{Z}$-objects: E.g. $F(I)$ is not a $\underline{Z}$-object , but $\operatorname{Irr} F(I)$ is order generating.

Note further that $\underline{\text{any}}$ dense subset of E is generating.

We have observed a counterexample to quite a few possible conjectures about order generation in coproducts. (In my notes onGK I had not yet understood this situation.) We return to the case of a family $\{T_j : j \in J\}$ in $\underline{CP}$. Now we assume that all T_j are subobjects of one and the same $T \in \underline{CP}$.

We have the following

LEMMA 7. Let $T_j \subseteq T$ in $\underline{CP}$, $j \in J$. Let $m : \coprod_J T_j \longrightarrow T$ be the canonical coproduct map in CP and $\hat{m} : T \longrightarrow \prod_J F_j$ its adjoint in CP^{op} (recall $\prod_{CPop} = \coprod_{CP}$!).
Define $X = \bigcup_J T_j \subseteq T$. Then the following statements are equivalent:

(1) m is surjective. (2) $\hat{m}$ is injective. (3) X is generating.

(4) $\bar{X}$ is generating. (5) $a = \inf ((\uparrow a)^{o} \cap X)$ for all $a \in A(T)$.
(6) $\bar{X}$ is order generating
If T is distributive, these conditions are equivalent to

~~[(6) $\bar{X}$ is order generating]~~ .(7) $IRR\ T \subseteq \bar{X}$.

If $T \in \underline{Z}$, (1)–(6) are equivalent to

(8) $Irr\ T \subseteq \bar{X}$. (9) $K(m) : K(T) \longrightarrow X_J\ K(T_j)$ is an embedding.

If J is finite, (1)–(6) are equivalent to

(10) X is order generating.

Proof. ~~x~~ The equivalence of ~~x~~ (3),(4),(5),(6) and (under the appropriate hypotheses) of (7),(8) was shown in Section 1. If J is finite, then $X = \bar{X}$, whence (10) <=>(6). The equivalence of (1) and (2) follows from ~~HM~~ ATLAS duality. ~~(2)~~ If we let $\tilde{X} \in \coprod_J T_j$ be the union of the images of T_j in the coproduct, then $m(\tilde{X}) = X$. Now ~~M~~ $\tilde{X}$ is generating in the coproduct, hence X is generating in im m. Thus (1)<=>(3).
(1) <=> (9) by HMS DUALITY [resp. (2)<=>(9) by ATLAS].

GK have observed that $\bar{X} = \bigcup \{T_j : j \in \bar{J}\}$ where the set of all T_j, $j \in \bar{J}$ is the closure in Sub(T) of ~~all~~ the set of all T_j, $j \in J$. Notice that the limit of a set of chains is a chain.

Up to this point, the investigation of GK and of JDL can be treated on the same background. In both cases one produces a closed generating set $X = \bigcup_J T_j$ of a ~~x~~ CP -object T (in fact both more or less restrict their attention to $\underline{Z}$) which is small in some sense. GK do this by finding a smart distributive closed sublattice D of Sub(T), $T \in \underline{Z}$, and by letting $\{T_j : j \in J\} = \overline{Irr}\ D$. JDL says: let us try to pick a small family $\{T_j : j \in J\}$ of chains if we can. A cardinal measure for the smallest number of chains doing the trick is what he calls the <u>spread</u> . I modify his definition somewhat:

DEFINITION 8. Let $T \in \underline{CP}$. The <u>spread</u> $SP(T)$ of T is the smallest cardinal a such that there is a family $T_j \in \text{Sub}\,(T)$ $j \in J$ of CHAINS T_j such that card $J = \underline{a}$, and that the equivalent conditions of Lemma 7 are satisfied.$\square$

For $\underline{Z}$-objects this means (by Lemma 7 , (8) and(10)) that Irr T is covered by the closure of the union of a collection of $\underline{a}$ chains $_j$,or equivalently, that the(discrete) character semilattice is a subsemilattice of a product of a collection of chains of cardinality $\underline{a}$, and that $\underline{a}$ is minimal w.r.t. this property. If $SP(T)$ is finite then under these circumstances (i.e. $T \in \underline{Z}$) $SP(T) = n$ means by Lemma 7, (8) that Irr T is covered by n chains in Sub(T) and that n is minimal w.r.t. this property. (This is more or less JDL's original definition.) From Lemma 7 we have immediately:

REMARK. 9. If the spread of a CP-object T is a finite number n, then n is the smallest natural number such that T is a quotient of a product of n- chains in CP and also the smallest number such that T can be embedded into a product of n chains <u>in CP^{op}</u> .

PROPOSITION 10. If $f : S \twoheadrightarrow T$ in CP , then $SP(T) \leq SP(S)$.
Proof. Let $X = \bigcup\{T_j : j \in J\}$, $T_j \in$ Sup S a chain with card J minimal and X generating. Then $f(X) = \bigcup\{f(T_j) : j \in J\}$ is generating and $f(T_j) \in$ Sub T is a chain.Hence the assertion$\square$.

EXAMPLE 11. Let $T = \{0,1,2\}^2$, $S = T \setminus \{(1,2)\}$. Then $SP(T) = 2$ $SP(S) = 3$, $S \subseteq T$ and T is a product of chains.$\square$

One observes immediately that $SP(\prod_J S_j) \leq \sum_J SP(S_j)$ from Lemma 7. From my experience with dimensional capacity I venture to say that equality holds. A proof may be difficult (it was with dimensional capacity).

Section 3. Breadth and spread (JDL)

(Sounds like bread and butter.) .

PROPOSITION 1 . Let $T \in CL$. Then br $T \leq SP(T)$, if $SP(T)$ is ~~finite~~ finite.

Proof. ~~xxx~~ There is an epic $e: \coprod_J C_j \longrightarrow T$ for card J = $SP(T)$ chains $C_j \in \mathrm{Sub}(T)$. Now br $T \leq$ br $\coprod_J C_j$ ~~= XXXXXXXXXXX(T)X~~

If J is finite, then br $\coprod_J C_j$ = br $\prod_J C_j = \sum_J$ br C_j = card J $= SP(T)$, since breadth is logarithmic.(i.e. br $\prod_J = \sum_J$ br).□

Let J be a set, and let $F(J)$ be the free object in $\underline{Z}$ hence in CP by Section 2 Prop. 5. We have $F(J) \cong c(\beta J)$ and $c(\beta J)$ contains the free discrete semilattice on the set βJ, hence br $F(J) \geq$ card $J = 2^{2^J}$, but SP $F(J) = J$. I suspect that br $T \leq 2^{2^{SP\ T}}$ remains correct in general.

LEMMA 2. (JDL). Let $S \in \underline{S}$ and suppose that $P \subseteq$ Prime S consists of mutually incomparable elements (i.e. P is an antichain). Let $F(P)$ be the free semilattice generated by P (in $\underline{S}$). Then the canonical map $F(P) \longrightarrow S$ is injective. In particuäar, card $P \leq$ br S.

Proof. We consider $F(P)$ as the $\cup$-semilattice of all finite subsets of P. Let $X, Y \in F(P)$ and suppose ~~‡~~ $\bigwedge X = \bigwedge Y$. If $y \in Y$, then $\bigwedge X \leq y$; since y is prime, there is an $x \in X$ with $x \leq y$. Since P is an antichain, we have $y = x \in X$. Theun $Y \subseteq X$. By symmetry $X \subseteq Y$.□

~~XXX~~PROPOSITION 3 (JDL). Let $T \in CL$ and $SP(T)$ finite. If T is distributive , then $SP(T)$ = br T.

Proof.(Indication! I do not quite understand Jimmie's proof.) By Prop. 1 , we must show $SP(T) \leq$ br T. We know IRR = PRIME hence PRIME $T \subseteq C_1 \cup \ldots \cup C_n$ for n chains C_k, n minimal (See Sect.2 Lemma 7, ~~s~~ condition (7)).

It appears to me that JDL concludes from this containment and minimality, that ~~k~~ PRIME T contains an antichain P of n elements. If this is so, then Lemma 2 shows n ~~x~~ $\leq$ br T.□