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TOPIC Representation of Colimites in <u>CL</u>, Part I

REFERENCE (1) Gierz, Hofmann, Keimel, Mislove: Relations with the interpolation property, extended version by Keimel, SCS-memo from 8/1/76

(2) Hofmann: The space of lower semicontinuous functions into a <u>CL</u>-object, SCS-Memo from 9/10/76

In (2) K.H.Hofmann gave a representation of copowers in <u>CL</u>. But this representation is false, because the following (wrong) assertion is used in the proof of lemma 2.4:

For every compact space S and every set J, $\beta(J\times S) = \beta J\times S$.

So we have to start again.

Let us recall some facts from (1):

I) Let <u>CSRIP</u> be the following category:

<u>Objects</u>: $(S, \sqsubset)$ where

(1) S is a complete lattice with smallest element 0.

(2) $\sqsubset$ is a binary relation on S such that

(2.1) (Interpolation property) $a \sqsubset b \Rightarrow \exists c \in S: a \sqsubset c \sqsubset b$

(2,2) $a \sqsubset b \Rightarrow a \leq b$

(2.3) $0 \sqsubset 0$

(2.4) $a \leq b \sqsubset c \Rightarrow a \sqsubset c$

(2.5) $a \sqsubset b \leq c \Rightarrow a \sqsubset c$

(2.6) $a \sqsubset c,\ b \sqsubset c \Rightarrow a \vee b \sqsubset c$

<u>Morphisms</u>: Let $(S, \sqsubset)$ and $(T, \sqsubset)$ be objects. A morphism between $(S, \sqsubset)$ and $(T, \sqsubset)$ is a mapping φ between S and

i.e. $a \sqsubset b$ implies $\varphi(a) \sqsubset \varphi(b)$.

Let $(S, \sqsubset)$ be an CSRIP-object. A subset $I \subseteq S$ is called an $\sqsubset$-ideal if $0 \in I$, if $a,b \in I$ implies $a \vee b \in I$, if $a \leq b \in I$ implies $a \in I$ and if for every $a \in I$ there exists an $b \in I$ such that $a \sqsubset b$. Every lattice ideal $I \subseteq S$ contains a largest $\sqsubset$-ideal denoted by $c(I)$. Denote by $P_{\sqsubset}(S)$ the set of all $\sqsubset$-ideals and by $P(S)$ the set of all lattice ideals. Then $c: P(S) \to P_{\sqsubset}(S)$ is a kernel operator and $P_{\sqsubset}(S)$ is a continuous lattice.

Furthermore, let $\varphi:(S, \sqsubset) \to (T, \sqsubset)$ be a CSRIP-morphism. Then $P_{\sqsubset}(\varphi):P_{\sqsubset}(T) \to P_{\sqsubset}(S)$; $I \mapsto c(\varphi^{-1}(I))$ is a CL-morphism, i.e. preserves arbitrary infima and updirected suprema. Therefore, $P_{\sqsubset}$:CSRIP $\to$ CL is a contravariant functor.

.) Let L,L' be continuous lattices and let $g:L \to L'$ be a map preserving arbitrary infima. Then its right adjoint $D(g):L' \to L$ preserves arbitrary suprema. Moreover, g is a CL-morphism iff for all $x,y \in L$, $x \ll y$ implies $D(g)(x) \ll D(g)(y)$ (see ATLAS, 1.19). Therefore D:CL $\longrightarrow$ CSRIP, $L \mapsto (L,\ll)$; $g \mapsto D(g)$ is a contravariant functor onto a full subcategory of CSRIP.

III) Theorem (see (1), 4.2): The functors P :CSRIP $\to$ CL and D:CL $\to$ CSRIP are adjoint on the left and $P_{\sqsubset} \circ D = 1_{CL}$. Especially, P and D both transfer limits to colimits.

The last theorem says that for the calculation of comlimites in CL it is very usefull to know the limits in CSRIP.

IV) Let Compl be the category of complete lattices with arbitrary sup-preserving maps as morphisms. It is well known that the forgetfull functor Compl $\to$ Set preserves limits. Let U:CSRIP $\to$ Compl be the forgetfull functor, $\underline{X}$ be a small category and $F:\underline{X} \to$ CSRIP be a diagramm in CSRIP. Then the limit of F in CSRIP may be calculated as follows:
Let $\varprojlim U \circ F$ be the limit of $U \circ F$ in Compl and for every $x \in |\underline{X}|$ let $pr_x:\varprojlim U \circ F \to U \circ F(x)$ be the canonical projection. Define a relation $\sqsubset$ on $\varprojlim U \circ F$ by $a \sqsubset b$ iff $pr_x(a) \sqsubset pr_x(b)$ in $(F(x), \sqsubset)$ for all $x \in |\underline{X}|$. It is easily checked that $\sqsubset$ satisfies (2.2)-(2.6). In (1) we constructed a largest relation $\blacktriangleleft$ contained in $\sqsubset$ which satisfies (2.1)-(2.6) in the following way: Let B denote the set of all dyadic rationals between 0 and 1 i.e. the set of all rational numbers $r = n/2^m$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, $n \leq 2^m$. Let $a,b \in \varprojlim U \circ F$. A dyadic chain from a to b is a map $\varphi:B \to \varprojlim U \circ F$ such that $r < s$ implies $\varphi(r) \sqsubset \varphi(s)$, $\varphi(0) = a$ and $\varphi(1) = b$. Define $a \blacktriangleleft b$ iff there is a dyadic chain from a to b.

V) Theorem: Let $F:\underline{X} \to \underline{CSRIP}$ be a diagram in CSRIP. Then $(\varprojlim U \circ F, \sqsubseteq)$ is the limit of F in CSRIP; the projections are the same as in Compl.

Proof: As $\sqsubseteq$ is coarser than $\sqsubset$ the projection $pr_x : \varprojlim U \circ F \to U \circ F(x)$ is a CSRIP-morphism. Let $(L, \sqsubset)$ be a CSRIP-object and let $(\varphi_x : L \to F(x))_{x \in |\underline{X}|}$ be a projective cone of CSRIP-morphisms. Then there exists a unique supremum-preserving $\varphi : L \to \varprojlim U \circ F$ such that $\varphi_x = pr_x \circ \varphi$ for all $x \in |\underline{X}|$. We have to show that φ preserves the additional relation $\sqsubset$. Firstly, note that for all $a, b \in L$, $a \sqsubset b$ implies $\varphi_x(a) = pr_x \circ \varphi(a) \sqsubset \varphi_x(b) = pr_x \circ \varphi(b)$ for all $x \in |\underline{X}|$ and hence $\varphi(a) \sqsubset \varphi(b)$ by the definition of $\sqsubset$ on $\varprojlim U \circ F$. But $a \sqsubset b$ in L implies that there exists a dyadic chain from a to b in L as L has the interpolation property. Hence there exists dyadic chain from $\varphi(a)$ to $\varphi(b)$ in $\varprojlim U \circ F$. Thus, $a \sqsubset b$ in L implies $\varphi(a) \sqsubseteq \varphi(b)$.

VI) Theorem: Let $F:\underline{X} \to \underline{CL}$ be a diagram in CL. Then the colimit $\varinjlim F$ is given by $P_{\sqsubset}(\varprojlim U \circ D \circ F, \sqsubseteq)$, the canonical injection $i_x : F(x) \to P_{\sqsubset}(\varprojlim U \circ D \circ F, \sqsubseteq)$ sends every $a \in F(x)$ to $c(\{f \in \varprojlim U \circ D \circ F \; ; \; pr_x(f) \ll a\})$.

VII) Corollary: Let L_j, $j \in J$, be continuous lattices. Then $\coprod L_j \cong P_{\sqsubset}(\prod L_j)$, where for $f, g \in \prod L_j$ we have $f \sqsubset g$ iff $f(j) \ll g(j)$ for all $j \in J$. The canonical injection $i_j : L_j \to \coprod L_j$ is given $i_j(a) = \{f \in \prod L_j \; ; \; f(j) \ll a \text{ and } f(k) \ll 1 \text{ for } j \neq k\}$.

In part II I will prove a representation of coproducts by upper semi-continuous sections in a "bundle" of continuous lattices and discuss some model theoretical properties of the "stalks" of that "bundle".