NAME(S)	Gerhard Gierz	DATE	M 11	D 11	Y 76

TOPIC Representations of Colimits in CL ; part II

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Beste Grüße!
Gerhard

Representations of Colimits in CL, part II

A) Bundles of Continuous Lattices

1) Prebundles and lower semicontinuous selections

1.1) Definition: A *prebundle of continuous lattice* is a triple $(\mathcal{E}, \rho, X)$ (written as $\rho: \mathcal{E} \to X$) such that

(i) $\mathcal{E}$ and X are compact spaces and $\rho: \mathcal{E} \to X$ is a continuous map from $\mathcal{E}$ onto X.

(ii) $\rho^{-1}(x)$ is a continuous lattice for every $x \in X$.

(iii) $\wedge : \operatorname{dom} \wedge = \{ (u,v) \in \mathcal{E} \times \mathcal{E} ; \ \rho(u) = \rho(v) \} \to \mathcal{E}$ is continuous.

1.2) Examples: (i) Let X be a compact space and L be a continuous lattice. Then $pr_1 : X \times L \to X$ is a prebundle of continuous lattices.

(ii) Let L_i, $i \in I$, be a family of lattices each of one having a smallest and a largest element. For every ultrafilter $\tilde{u}$ on I let $L_{\tilde{u}}$ be the ultraproduct of the L_i given by $\tilde{u}$ and $\pi_{\tilde{u}} : \prod L_i \to L_{\tilde{u}}$ be the canonical homomorphism. Then every $\pi_{\tilde{u}}$ gives an embedding $\hat{\pi}_{\tilde{u}} : \hat{L}_{\tilde{u}} \rightarrowtail \widehat{\prod L_i}$ via the Pontryagin duality for discrete and boolean semilattices, where $\hat{L}$ denotes the lattice of all filters of L. In [7] we have shown that $\bigcup_{\tilde{u} \in \beta I} \hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}})$ is closed in the Lawson topology of $\widehat{\prod L_i}$. I now show $\hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}}) \cap \hat{\pi}_{\tilde{v}}(\hat{L}_{\tilde{v}}) = \{ \widehat{\prod L_i} \}$ for $\tilde{u} \neq \tilde{v}$. Indeed, let $F \in \hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}}) \cap \hat{\pi}_{\tilde{v}}(\hat{L}_{\tilde{v}})$. Then there exist two filters $F_{\tilde{u}} \in \hat{L}_{\tilde{u}}$ and $F_{\tilde{v}} \in \hat{L}_{\tilde{v}}$ such that $\pi_{\tilde{u}}^{-1}(F_{\tilde{u}}) = \pi_{\tilde{v}}^{-1}(F_{\tilde{v}}) = F$. Choose an $M \subseteq I$ such that $M \in \tilde{u}$ and $I \setminus M \in \tilde{v}$ and define two elements $a, b \in \prod L_i$ by $a(i) = 1$, $b(i) = 0$ for $i \in M$ and $a(i) = 0$, $b(i) = 1$ for $i \in I \setminus M$. Then $\pi_{\tilde{u}}(a) = \pi_{\tilde{u}}(1) \in F_{\tilde{u}}$ and $\pi_{\tilde{v}}(b) = \pi_{\tilde{v}}(1) \in F_{\tilde{v}}$, hence $a, b \in F$ and so is $a \wedge b$. But this implies $F = \widehat{\prod L_i}$ and hence the $\hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}})$ intersect pairwise only in the open closed set $\{ \prod L_i \}$.

Now define $\mathcal{E} := \bigcup_{\tilde{u} \in \beta I} \{\tilde{u}\} \times \hat{L}_{\tilde{u}}$. As just shown, the mapping $(\tilde{u}, F) \mapsto \pi_{\tilde{u}}^{-1}(F)$: $\mathcal{E} \setminus \{ (\tilde{u}, L_{\tilde{u}}) ; \tilde{u} \in \beta I \} \to \bigcup_{\tilde{u} \in \beta I} \hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}}) \setminus \{\prod L_i\}$ is a bijection onto the compact set

$\bigcup_{\tilde{u}\in\beta I} \hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}}) \setminus \{\Pi L_i\}$. Endow $\mathcal{E} \setminus \{(\tilde{u}, L_{\tilde{u}});\ \tilde{u}\in\beta I\}$ with the topology induced by that mapping and endow $\{(\tilde{u}, L_{\tilde{u}});\ \tilde{u}\in\beta I\}$ with the topology induced by βI and finally, endow $\mathcal{E}$ with the sum topology of these two topologies.

Then $p: \mathcal{E}\to\beta I$ is a prebundle of continuous lattices, where p denotes the obvious map: Clearly, $\mathcal{E}$ and βI are compact spaces. Furthermore, $p: \mathcal{E}\to\beta I$ is continuous: Let $A\in\beta I$ be closed and $\mathcal{A}(\widehat{\Pi L_i})$ be the hyperspace of all compact subsets of $\widehat{\Pi L_i}$. In [ ] we have shown that the mapping $\tilde{u}\mapsto \hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}})$: $: \beta(I) \to \mathcal{A}(\widehat{\Pi L_i})$ is continuous. Therefore $\{\hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}});\ \tilde{u}\in A\}$ is closed in $\mathcal{A}(\widehat{\Pi L_i})$ and hence $\bigcup_{\tilde{u}\in A}(\hat{\pi}_{\tilde{u}}(\hat{L}_{\tilde{u}}))$ is closed in $\widehat{\Pi L_i}$. But this implies the closeness of $p^{-1}(A) =$ $(\bigcup_{\tilde{u}\in A}(\{\tilde{u}\}\times\hat{L}_{\tilde{u}}) \setminus \{(\tilde{u}, L_{\tilde{u}});\ \tilde{u}\in\beta I\}) \cup \{(\tilde{u}, L_{\tilde{u}});\ \tilde{u}\in A\}$ in $\mathcal{E}$. Moreover, $p^{-1}(\tilde{u}) = \{\tilde{u}\}\times\hat{L}_{\tilde{u}}$ is a continuous lattice isomorphic with $\hat{L}_{\tilde{u}}$ (Note that $\{L_{\tilde{u}}\}\subseteq\hat{L}_{\tilde{u}}$ is open closed in the Lawson topology). The continuity of $\wedge: \{(u,v)\in\mathcal{E}\times\mathcal{E};\ p(u)=p(v)\}\to\mathcal{E}$ follows easily from the continuity of $\wedge: (\widehat{\Pi L_i})^2 \to \widehat{\Pi L_i}$.

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1.3) <u>Definition</u>: The prebundle constructed in example 1.2) (ii) is called the <u>bundle associated with the direct product of the</u> L_i.

.4) <u>Proposition</u>: The Lawson topology coincides with the induced topology on each stalk.

Proof. The Lawson topology is the unique compact topology such that the operation $\wedge$ is continuous.

1.5) <u>Proposition</u>: The graph of „$\leq$", i.e. the set $\{u,v\in\mathcal{E}\times\mathcal{E};\ p(u)=p(v) \text{ and } u\leq v\}$ is closed

Proof. Easy calculation using the continuity of $\wedge$.

1.6) <u>Definition</u>: A selection $\sigma: X\to\mathcal{E}$ is called <u>lower semicontinuous</u> if it's upper graph $G(\sigma) = \{u;\ u\geq \sigma p(u)\}$ is closed

If $U\subseteq X$ is open, then every continuous selection $\sigma: U\to\mathcal{E}$ is called a <u>local section over</u> U.

Denote by

$LC(p)$ the set of all lower semicontinuous selections.

$\Gamma(p)$ the set of all global sections.

1.7) Remark (i) Proposition 1.5 implies $\Gamma(p) \subseteq LC(p)$.

(ii) $LC(p)$ is not empty as $\mathbb{O} : x \mapsto 0_x : X \to \mathcal{E}$ is lower semicontinuous.

1.8) Proposition: (see [4]) T.A.E

(i) $\sigma : X \to \mathcal{E}$ is lower semicontinuous.

(ii) $\bigcap \{ \overline{\sigma(\mathcal{U})} \, ; \, \mathcal{U} \in \mathcal{U}(x) \} \subseteq \uparrow \sigma(x)$ for all $x \in X$, where $\mathcal{U}(x)$ denotes the filter of all open neighborhoods of x.

(iii) For every convergent net $(x_i)_{i\in I}$ of X with $\lim x_i = x$ and every cluster point u of $(\sigma(x_i))_{i\in I}$ we have $u \geq \sigma(x)$.

Proof. (i) $\Rightarrow$ (ii): First, note that $p\left(\bigcap \{ \overline{\sigma(\mathcal{U})} \, ; \, \mathcal{U} \in \mathcal{U}(x) \}\right) \subseteq \bigcap \{ p(\overline{\sigma(\mathcal{U})}) \, ; \, \mathcal{U} \in \mathcal{U}(x) \} = \bigcap \{ \overline{\mathcal{U}} \, ; \, \mathcal{U} \in \mathcal{U}(x) \} = \{x\}$. Hence $\bigcap \{ \overline{\sigma(\mathcal{U})} \, ; \, \mathcal{U} \in \mathcal{U}(x) \} \subseteq G(\sigma) \cap p^{-1}(x) = \uparrow \sigma(x)$ by the definition of lower semicontinuity.

(ii) $\Rightarrow$ (iii): Clear, as every cluster point of $(\sigma(x_i))_{i\in I}$ is contained in $\bigcap \{ \overline{\sigma(\mathcal{U})} \, ; \, \mathcal{U} \in \mathcal{U}(x) \}$, provided that $(x_i)_{i\in I}$ converges towards x.

(iii) $\Rightarrow$ (i): Let $(u_i)_{i\in I}$ be a convergent net of $G(\sigma)$ and $u = \lim u_i$. Let $x_i := p(u_i)$ and $x = p(u)$. Then $(x_i)_{i\in I}$ converges to x. If k is a cluster point of $(\sigma(x_i))_{i\in I}$, then proposition 1.4 and $\sigma(x_i) \leq u_i$ implies $k \leq u$. Hence $\sigma(x) \leq k \leq u$ by (iii).

1.9) Definition: A subset $A \subseteq \mathcal{E}$ is called a prebundle filter provided that

(i) A is closed

(ii) $p^{-1}(x) \cap A$ is a non empty filter of $p^{-1}(x)$.

Denote by $BF(p)$ the complete lattice of all prebundle filters, ordered by inclusion.

1.10) Remark: Every prebundle filter A gives a lower semicontinuous selection σ_A by $\sigma_A(x) := \inf(p^{-1}(x) \cap A)$. Conversely if σ is a lower semicontinuous selection, then $G(\sigma)$

is a prebundle filter. Note that $G(\sigma_A) = A$ and $\sigma = \sigma_{G(\sigma)}$. Hence we have proved

1.11) <u>Proposition</u>: Let $\mathcal{F} \subseteq LC(p)$ be a family of lower semicontinuous selections. Then $\sup \mathcal{F}$ with $(\sup \mathcal{F})(x) := \sup \mathcal{F}(x)$ is lower semicontinuous. Especially, $LC(p)$ is a complete lattice dually isomorphic with $BF(p)$.

1.12) <u>Proposition</u>: Let $\sigma, \tau \in LC(p)$. Then $\sigma \wedge \tau \in LC(p)$, where $\sigma \wedge \tau(x) := \sigma(x) \wedge \tau(x)$. Moreover, if both σ and τ are continuous, then so is $\sigma \wedge \tau$.

Proof. $G(\sigma \wedge \tau) = \uparrow \{ u \wedge u' ;\ p(u) = p(u') \text{ and } u \geq \sigma p(u),\ u' \geq \sigma p(u') \} = \uparrow (G(\sigma) \wedge G(\tau))$. But $G(\sigma) \wedge G(\tau)$ is closed by the continuity of $\wedge$ and hence $\uparrow (G(\sigma) \wedge G(\tau))$ is closed. If σ and τ are continuous, then $\sigma \wedge \tau$ is continuous by the continuity of $\wedge$.

1.13) <u>Notation</u>: If $\mathcal{U} \subseteq X$ is open, then the <u>characteristic function</u> $\chi_{\mathcal{U}}$ of $\mathcal{U}$ is defined by $\chi_{\mathcal{U}}(x) = 1_x$ if $x \in \mathcal{U}$ and $\chi_{\mathcal{U}}(x) = 0_x$ if $x \notin \mathcal{U}$.

I do not know, whether or not $\chi_{\mathcal{U}}$ is lower semicontinuous in general, but

1.14) <u>Proposition</u>: (i) Let $\mathcal{U} \subseteq X$ be open and $\sigma \in LC(p)$ be lower semicontinuous. Then $\sigma \wedge \chi_{\mathcal{U}}$ is lower semicontinuous.

(ii) If $\mathbb{1} : X \to \mathcal{E},\ x \mapsto 1_x$ is lower semicontinuous, then so is the characteristic function of every open subset $\mathcal{U} \subseteq X$.

Proof. (i) $G(\sigma \wedge \chi_{\mathcal{U}}) = p^{-1}(X \setminus \mathcal{U}) \cup G(\sigma)$

(ii) $\chi_{\mathcal{U}} = \mathbb{1} \wedge \chi_{\mathcal{U}}$.

2. Bundles of Continuous lattices

2.1) Definition: A prebundle $p: \mathcal{E} \to X$ is called a quasibundle, if it satisfies

(iv) For every pair $u, u' \in \mathcal{E}$ such that $p(u) = p(u')$ and $u \ll u'$ there exist an open neighborhood $\mathcal{U}$ of $p(u)$ and a lower semicontinuous selection $\sigma \in LC(p)$ satisfying $u \leq \sigma p u \ll u'$ and $\{v \in \mathcal{E};\ p(v) \in \mathcal{U} \text{ and } v \gg \sigma p(v)\}$ is open.

A prebundle $p: \mathcal{E} \to X$ is called a bundle, if it satisfies

(iv') For every pair $u, u' \in \mathcal{E}$ satisfying $p(u) = p(u')$ and $u \ll u'$ there exist an open neighborhood $\mathcal{U}$ of $p(u)$ and a local section $\sigma: \mathcal{U} \to \mathcal{E}$ such that $u \leq \sigma p(u) \ll u'$ and $\{v \in \mathcal{E};\ p(v) \in \mathcal{U} \text{ and } v \gg \sigma p(v)\}$ is open.

2.2) Proposition: Every bundle is a quasibundle.

Proof. If $\sigma: \mathcal{U} \to \mathcal{E}$ is a local section, then $\chi_{\mathcal{U}} \wedge \sigma$ is defined globally and a lower semicontinuous selection.

2.3) Proposition: If $p: \mathcal{E} \to X$ is a bundle, then $p: \mathcal{E} \to X$ is open.

Proof. Let $O \subseteq \mathcal{E}$ be open and $x \in p(O)$. Choose an $u \in p^{-1}(x) \cap O$. Then there exists an $u' \in p^{-1}(x) \cap O$ such that $u' \ll u$ and $\Uparrow u' \cap \downarrow u \subseteq O$. Therefore, using the interpolation property, we may find an open neighborhood $\mathcal{U}$ of x and a local continuous section $\sigma: \mathcal{U} \to \mathcal{E}$ satisfying $u' \ll \sigma p(u) \ll u$. But then $\sigma(x) \in O$. Hence there exists an open neighborhood $x \in \mathcal{V} \subseteq \mathcal{U}$ such that $\sigma(\mathcal{V}) \subseteq O$. But this implies $x \in \mathcal{V} \subseteq p(O)$.

2.4) Examples: (i) The prebundle defined in example 1.2 (i) is a bundle, because the constant selections are global sections and as $\{(x,u) \in X \times L;\ u \gg v\} = X \times \Uparrow v$ is open in $X \times L$.

(ii) The bundle associated with the direct product of lattices L_i is indeed a bundle in the sense of definition 2.1. To prove this statement, we need

2.5) <u>Lemma</u>: Let $\tilde{u}$ be an ultrafilter on I and $F_i \in \hat{L}_i$ for all $i \in I$. Define $F_{\tilde{u}} := \pi_{\tilde{u}}(\prod F_i) \in \hat{L}_{\tilde{u}}$. Then $\lim_{\tilde{u}} \hat{\pi}_i(F_i) = \hat{\pi}_{\tilde{u}}(F_{\tilde{u}})$

Proof. Let O be an open neighborhood of $\hat{\pi}_{\tilde{u}}(F_{\tilde{u}})$ of the form $O(c) = \{F \in \hat{\prod} L_i ;\ c \in F\}$. Then $\pi_{\tilde{u}}(c) \in F_{\tilde{u}} = \pi_{\tilde{u}}(\prod L_i)$. Then $M := \{i ;\ c(i) \in F_i\}$ is contained in $\tilde{u}$. But this is equivalent to $\{i ;\ c \in \hat{\pi}_i(F_i)\} \in \tilde{u}$. Hence we have found an $M \in \tilde{u}$ such that $\{\hat{\pi}_i(F_i) ;\ i \in M\} \subseteq O(c)$.
If O is of the form $O(\neg c) := \{F \in \hat{\prod} L_i ;\ c \notin F\}$ we have $\pi_{\tilde{u}}(c) \notin F_{\tilde{u}}$ and hence $M := \{i ;\ c(i) \notin F_i\} = \{i ;\ c \notin \hat{\pi}_i(F_i)\} \in \tilde{u}$. But this implies $\{\hat{\pi}_i(F_i) ;\ i \in M\} \subseteq O(\neg c)$.
As the sets $O(c)$ and $O(\neg c)$ form a subbase of the topology of $\hat{\prod} L_i$, we are done.

Now let (w, F_w) and (w, G_w) be elements of $\mathcal{E}$ such that $(w, F_w) \ll (w, G_w)$.
Then there exists an $a \in \prod L_i$ with the property that $F_w \subseteq \uparrow \pi_w(a)$ and $\pi_w(a) \in G_w$.
Define a selection $\sigma_a : \beta I \to \mathcal{E}$ by $\sigma_a(\tilde{u}) = (\tilde{u}, \uparrow \pi_{\tilde{u}}(a))$. Then

(*) σ_a is continuous

(**) $\Uparrow \sigma_a$ is open

(***) $(w, F_w) \le \sigma_a(w) \ll (w, G_w)$.

Proof of (*). Let $M := \{i \in I ;\ 0_i \in \uparrow \pi_i(a)\}$, $A := \{\tilde{u} \in \beta I ;\ M \in \tilde{u}\}$ and $B := \{\tilde{u} \in \beta I,\ M \notin \tilde{u}\}$.
Then A and B are clopen and $A \cup B = \beta I$. For all $\tilde{u} \in A$ we have $\sigma_a(\tilde{u}) = (\tilde{u}, L_{\tilde{u}})$ and hence the restriction of σ_a to A is continuous by the definition of the topology on $\{(\tilde{u}, L_{\tilde{u}}) ;\ \tilde{u} \in \beta I\} \subseteq \mathcal{E}$. If $\tilde{u} \in B$, then $\sigma_a(\tilde{u}) \neq (\tilde{u}, L_{\tilde{u}})$ and hence the restriction of σ_a to B is continuous by Lemma 2.5 and the definition of the topology on $\mathcal{E} \setminus \{(\tilde{u}, L_{\tilde{u}}) ;\ \tilde{u} \in \beta I\}$.

Proof of (**). Let A and B be as in (*). Then $\Uparrow \sigma_a = E_1 \cup E_2$ where
$E_1 = \{(\tilde{u}, L_{\tilde{u}}) ;\ \tilde{u} \in \beta I\} \cup \{(\tilde{u}, F_{\tilde{u}}) ;\ \tilde{u} \in B,\ F_{\tilde{u}} \neq L_{\tilde{u}} \text{ and } F_{\tilde{u}} \supseteq \uparrow \pi_{\tilde{u}}(a)\}$ (Recall that the filters of the form $\uparrow \pi_{\tilde{u}}(a)$ are compact elements of $\hat{L}_{\tilde{u}}$ and that $L_{\tilde{u}} = \uparrow 0$ itself is a compact element, so that $\Uparrow(\uparrow \pi_{\tilde{u}}(a)) = \{F_{\tilde{u}} \in \hat{L}_{\tilde{u}} ;\ \uparrow \pi_{\tilde{u}}(a) \subseteq F_{\tilde{u}}\}$!) But $F_{\tilde{u}} \supseteq \uparrow \pi_{\tilde{u}}(a)$ is equivalent to $a \in \hat{\pi}_{\tilde{u}}(F_{\tilde{u}})$ and hence E_1 and E_2 are open in $\mathcal{E}$.

Proof of (***). Clear!

2.6) <u>Proposition</u>: Let $p: \mathcal{E} \to X$ be a quasibundle of continuous lattices. Then

(i) Through every point passes a lower semicontinuous selection

(ii) $\mathbb{1}: X \to \mathcal{E}$ is continuous

(iii) $\chi_{\mathcal{U}}$ is lower semicontinuous for every open subset $\mathcal{U} \subseteq X$.

Proof. (i) follows from (iv) of definition 2.1 and proposition 1.11.

(ii) $\mathbb{1} = \sup LC(p)$ by (i), hence $\mathbb{1}$ is lower semicontinuous. Let $(x_i)_{i\in I}$ be a convergent net of X with $x = \lim x_i$. By proposition 1.8, 1_x is the unique cluster point of $(1_{x_i})_{i\in I}$ and hence $1_x = \lim 1_{x_i}$. This yields the continuity of $\mathbb{1}$.

(iii) Clear by proposition 1.14.

2.7) <u>Proposition</u> If $p: \mathcal{E} \to X$ is a bundle of continuous lattices, then $O: X \to \mathcal{E}$ is a global continuous section.

Proof. Let $(x_i)_{i\in I}$ be a convergent net of X with limit point x and let k be a cluster point of $(O_{x_i})_{i\in I}$. Assume that $k \neq O_x$. Then by (iv') of definition 2.1 we find a local continuous section $\sigma: \mathcal{U} \to \mathcal{E}$ such that $O \le \sigma(x) \ll O < k$ which is impossible as $\tau(x_i) \ge O_{x_i}$ for all $i \ge i_0$ and therefore $k' \le \tau(x) = O_x$ for every cluster point k' of $(O_{x_i})_{i\in I}$.

2.8) <u>Corollary</u>: If $p: \mathcal{E} \to X$ is a bundle of continuous lattice and $\mathcal{U} \subseteq X$ is an clopen subset, then $\chi_{\mathcal{U}}: \mathcal{E} \to X$ is a continuous section.

2.9) <u>Theorem</u>: Let $p: \mathcal{E} \to X$ be a quasibundle of continuous lattices. Then $LC(p)$ is a continuous lattice. In this continuous lattice, $\sigma \ll \tau$ is equivalent to $G(\tau) \subseteq G(\sigma)^{o}$, where ^{o} denotes the open kernel.

Moreover, if $p: \mathcal{E} \to X$ is a bundle and if X is totally disconnected, then every lower semicontinuous selection is the supremum of all continuous sections way below it and $\sigma \ll \tau$ is equivalent to the existence of an $\sigma' \in T(p)$ such that $\sigma \le \sigma' \ll \tau$.

Proof. $LC(p)$ is a complete lattice, so for showing that $LC(p)$ is a continuous lattice it is enough to show that every $\sigma \in LC(p)$ is the supremum of all lower semicontinuous selections (resp. sections if X is totally disconnected) way below it. First note that $G(\tau) \subseteq G(\sigma)^\circ$ implies $\sigma \ll \tau$. Indeed, let $\tau_i \in LC(p)$ be updirected and $\tau \leq \sup \tau_i$. Then $G(\sigma)^\circ \supseteq G(\tau) \supseteq \bigcap G(\tau_i)$. Because $\mathcal{E}$ is compact, there is an τ_i such that $G(\tau_i) \subseteq G(\sigma^\circ)$ and hence $\sigma \leq \tau_i$. — Now let $\tau \in LC(p)$, $x \in X$ and $k \in p^{-1}(x)$ such that $k \ll \tau(x)$. We have to construct a $\sigma \in LC(p)$ (resp. $\sigma \in \Gamma(p)$ if X is totally disconnected) fulfilling $\sigma \ll \tau$ and $k \leq \sigma(x)$. This σ will have the additional property that $G(\tau) \subseteq G(\sigma)^\circ$. First, (iv) (resp. iv') of definition 1.2 implies the existence of an open neighborhood $\mathcal{W}$ of x and a lower semicontinuous selection σ' (resp. local section $\sigma' : \mathcal{W} \to \mathcal{E}$) such that $\mathcal{O} := \{u \in \mathcal{E};\ p(u) \in \mathcal{W} \text{ and } u \gg \sigma' p(u)\}$ is open and such that $k \ll \sigma'(x) \ll \tau(x)$. Hence by proposition 1.8 we have $\bigcap \{\overline{\tau(\mathcal{U})};\ \mathcal{U} \in \mathcal{V}(x)\} \subseteq \mathcal{O}$. The compactness of $\mathcal{E}$ yields a neighborhood $\mathcal{V} \subseteq \mathcal{W}$ of x satisfying $\overline{\tau(\mathcal{V})} \subseteq \mathcal{O}$. Choose an (clopen, if X is totally disconnected) neighborhood $\mathcal{U}$ of x with $\overline{\mathcal{U}} \subseteq \mathcal{V}$. Then for $\sigma := \chi_{\mathcal{U}} \wedge \sigma'$ (which is defined on all of X and continuous, if X is totally disconnected) we have $G(\tau) \subseteq G(\sigma)^\circ$ and $k \ll \sigma(x) \ll \tau(x)$. Indeed, $k \ll \sigma'(x) = \sigma(x)$ and $G(\tau) \subseteq \mathcal{O} \cup p^{-1}(X \setminus \overline{\mathcal{U}}) \subseteq G(\sigma)^\circ$. Thus we have shown that $\tau = \sup\{\sigma \in LC(p);\ G(\tau) \subseteq G(\sigma)^\circ\}$ (resp. $\tau = \sup\{\sigma \in \Gamma(p);\ G(\tau) \subseteq G(\sigma)^\circ\}$). Hence $LC(p)$ is a continuous lattice. — Last, let $\sigma \ll \tau$. Then we may find $\sigma_1, \dots, \sigma_n$ such that $\sigma \ll \sigma_1 \vee \dots \vee \sigma_n$ and $G(\tau) \subseteq G(\sigma_1)^\circ, \dots, G(\sigma_n)^\circ$. But $\sigma \leq \sigma_1 \vee \dots \vee \sigma_n$ implies $G(\sigma_1 \vee \dots \vee \sigma_n) = G(\sigma_1) \cap \dots \cap G(\sigma_n) \subseteq G(\sigma)$ and hence $G(\tau) \subseteq G(\sigma_1)^\circ \cap \dots \cap G(\sigma_n)^\circ \subseteq G(\sigma)^\circ$. — If X is totally disconnected, we have to construct an $\sigma' \in \Gamma(p)$ such that $\sigma \leq \sigma' \ll \tau$ is fulfilled. First, choose an $\tau' \in LC(p)$ such that $\sigma \ll \tau' \ll \tau$. Then we have $G(\tau) \subseteq G(\tau')^\circ$ and $G(\tau') \subseteq G(\sigma)^\circ$. Let $x \in X$ be a point. As shown above we may find a clopen neighborhood $\mathcal{U}_x$ of x and a continuous section $\sigma''_x : X \to \mathcal{E}$ vanishing outside of $\mathcal{U}_x$ and satisfying $G(\tau) \subseteq G(\sigma''_x)^\circ$ as well as $\sigma''_x(x) \in G(\tau') \subseteq G(\sigma)^\circ$. The continuity of σ''_x yields the existence of an open closed neighborhood $\mathcal{V}_x$ of x such that $\sigma''_x(y) \in G(\sigma)^\circ$ for all $y \in \mathcal{V}_x$. Define $\sigma'_x := \chi_{\mathcal{V}_x} \wedge \sigma''_x$. Then σ'_x is continuous (see 1.12 and 2.8) and we still have $\sigma'_x(y) \in G(\sigma)^\circ$ for all $y \in \mathcal{V}_x$ and $G(\tau) \subseteq G(\sigma'_x)^\circ$. X being compact we may find finitely many $x_1, \dots, x_n \in X$ such that $X = \mathcal{V}_{x_1} \cup \dots \cup \mathcal{V}_{x_n}$. Let $\mathcal{W}_1, \dots, \mathcal{W}_n$ be a cover of X consisting of pairwise disjoint clopen sets such that $\mathcal{W}_i \subseteq \mathcal{V}_{x_i}$, $i = 1, \dots, n$, and define $\sigma_i := \sigma'_{x_i} \wedge \chi_{\mathcal{W}_i}$. Clearly, $G(\tau) \subseteq G(\sigma'_{x_i})^\circ \subseteq G(\sigma_i)^\circ$ for all i. Moreover,

Moreover, for every $x \in X$ there exists an i such that $x \in W_i \subseteq V_{x_i}$.
Hence $\sigma(x) \leq \chi_{W_i} \wedge \sigma'_{x_i}(x) = \sigma_i(x)$ for that i. This implies $\sigma \leq \sigma_1 \vee \ldots \vee \sigma_n \ll \tau$.
But $\sigma' := \sigma_1 \vee \ldots \vee \sigma_n$ is continuous, because $\sigma'/_{W_i} = \sigma_i/_{W_i} = \sigma_{x_i}/_{W_i}$.

Now let $p : \mathcal{E} \to X$ be a quasibundle and $u \in \mathcal{E}$ be an element. Define $\eta_u : X \to \mathcal{E}$ by $\eta(p(u)) = u$ and $\eta(x) = 1_x$ for $x \neq p(u)$. Clearly, η_u is lower semicontinuous

2.10) <u>Theorem</u> Let $p : \mathcal{E} \to X$ be a quasibundle. Then

(i) The mapping $\eta_- : \mathcal{E} \longrightarrow LC(p)$ is continuous

(ii) For every $x \in X$ the restriction $\eta_- : p^{-1}(x) \to LC(p)$ is a <u>CL</u>-morphism.

Proof. (i) Let $\tilde{u}$ be an ultrafilter on $\mathcal{E}$ and $u = \lim \tilde{u}$. We have to show that $\eta_u = \eta_{\lim \tilde{u}} = \lim_{\tilde{u}} \eta_v = \sup_{M \in \tilde{u}} \inf_{v \in M} \eta_v$. Define γ to be the right hand side of this equation. Let $k \in \mathcal{E}$ be way below $\tilde{u}$. Then, by (iv) of definition 2.1, there exists an arbitrary small neighborhood $\mathcal{U}$ of $p(u)$ and a lower semicontinuous selection σ such that $k \ll \sigma(p(u)) \ll u$ and such that $\mathcal{O} := \{w \in \mathcal{E} ;\ p(w) \in \mathcal{U} \text{ and } w \gg \sigma(p(w))\}$ is open in $\mathcal{E}$. Choose an open neighborhood $\mathcal{V}$ of $p(u)$ such that $\overline{\mathcal{V}} \subseteq \mathcal{U}$. As $\mathcal{O}$ and $p^{-1}(\overline{\mathcal{V}})$ are neighborhoods of u we ·y find an $M \in \tilde{u}$ such that $M \subseteq \mathcal{O} \cap p^{-1}(\overline{\mathcal{V}})$. But this implies $\inf_{v \in M} \eta_v \geq \sigma \vee \chi_{X \setminus \overline{\mathcal{V}}}$ and hence $\gamma \geq \sigma \vee \chi_{X \setminus \overline{\mathcal{V}}}$. Therefore we have shown $\gamma \geq \sup_{\mathcal{V} \in \mathcal{U}(p(u))} \sigma \vee \chi_{X \setminus \overline{\mathcal{V}}} = \eta_{\sigma(p(u))} \geq \eta_k$
As suprema in $LC(p)$ are calculated pointwise this yields $\gamma \geq \eta_u$. Therefore $\gamma = \eta_{u'}$ for an $u \leq u' \in p^{-1}p(u)$. Assume, if possible, that $u' \neq u$. Then there exists a lower semicontinuous selection σ satisfying $\sigma \ll \eta_{u'}$ and $\sigma \not\leq \eta_u$. Hence, we find an $M \in \tilde{u}$ such that $\sigma \ll \inf_{v \in M} \eta_v$. But this implies $v \in G(\sigma)$ for all $v \in M$ and consequently, $u = \lim \tilde{u} \in \overline{G(\sigma)} = G(\sigma)$, a contradiction to $\sigma(p(u)) \not\leq \eta_u(p(u)) = u$.
(ii) Clear by (i).

2.11) <u>Theorem</u> Let $p: \mathcal{E} \to X$ be a bundle over a totally disconnected base space X. If A is a dense subspace of X, then $LC(p)$ is generated by $\eta_-(p^{-1}(A))$.

Proof. By 2.9. it suffices to show that every continuous section $\sigma \in \Gamma(p)$ may be represented in the form $\sigma = \inf\{\eta_{\sigma(a)} ; a \in A\}$. Indeed, σ is a lower bound of $\{\eta_{\sigma(a)} ; a \in A\}$ and proposition 1.8 implies, that σ is the greatest lower bound.

2.12) <u>Theorem:</u> Let $(L_i)_{i \in I}$ be lattices with 0 and 1 and let $p: \mathcal{E} \to \beta I$ be the bundle associated with the direct product of the L_i. Then $LC(p)$ is isomorphic with the lattice of all filters of $\prod L_i$. The (canonical) isomorphism $\sigma_-: \widehat{\prod L_i} \to LC(p)$ is given by $F \mapsto \sigma_F$, where $\sigma_F(\tilde{u}) := (\tilde{u}, \pi_{\tilde{u}}(F))$ for all $\tilde{u} \in \beta I$. Moreover, if $F = \uparrow a$ is the principal filter generated by $a \in \prod L_i$, then $\sigma_{\uparrow a} = \sigma_a$ is a continuous section such that $\{u ; u \gg \sigma_a p(u)\}$ is open. Furthermore, a selection σ is continuous if and only if $\sigma = \sigma_F$, where F is a product of filters $F_i \in \hat{L}_i$. Hence $\Gamma(p) \cong \prod \hat{L}_i$.

Proof. Let F be a filter of $\prod L_i$. Then $G(\sigma_F) = \{(\tilde{u}, G_{\tilde{u}}) ; G_{\tilde{u}} \neq L_{\tilde{u}}$ and $\pi_{\tilde{u}}(F) \subseteq G_{\tilde{u}}\} \cup \{(\tilde{u}, L_{\tilde{u}}) ; \tilde{u} \in \beta I\} =$
$= \{(\tilde{u}, G_{\tilde{u}}) ; \pi_{\tilde{u}}^{-1}(G_{\tilde{u}}) \neq \prod L_i$ and $F \subseteq \pi_{\tilde{u}}^{-1}(G_{\tilde{u}})\} \cup \{(\tilde{u}, L_{\tilde{u}}) ; \tilde{u} \in \beta I\}$. But $\{G \in \widehat{\prod L_i} ; G \neq \widehat{\prod L_i}$ and $F \subseteq G\}$ is closed in $\widehat{\prod L_i}$ and hence $\{(\tilde{u}, G_{\tilde{u}}) ; \pi_{\tilde{u}}^{-1}(G_{\tilde{u}}) \neq \prod L_i$ and $F \subseteq \pi_{\tilde{u}}^{-1}(G_{\tilde{u}})\}$ is closed in $\mathcal{E} \setminus \{(\tilde{u}, L_{\tilde{u}}) ; \tilde{u} \in \beta I\}$ by the definition of the topology on $\mathcal{E}$. Therefore $G(\sigma_F)$ is closed and σ_F is a lower semicontinuous selection. Next, we prove the last claim of the theorem. Indeed, if F is the product of filters $F_i \in \hat{L}_i$, then a look on the proof of $(*)$ in example 2.4 (ii) shows that σ_F is continuous. Conversely, if $\sigma: \beta I \to \mathcal{E}$ is continuous, define $F_i := \sigma(i)$ for all $i \in I$ and $F := \prod F_i$. Then σ_F and σ agree on the dense set $I \subseteq \beta I$, hence they are equal. Thus we have shown that $\Gamma(p) \subseteq \operatorname{im} \sigma_-$. But every lower semicontinuous selection is a updirected supremum of continuous one's by theorem 2.9. Because σ_- preserves updirected suprema and because for product filters $F = \prod F_i$ and $G = \prod G_i$ we have $\sigma_F \leq \sigma_G$ iff $F \subseteq G$, the mapping σ_- is surjective. Hence the theorem is proved if we can show that $F \subseteq G$ is equivalent to $\sigma_F \leq \sigma_G$ for all $F, G \in \widehat{\prod L_i}$. Clearly, $F \subseteq G$ implies $\sigma_F(\tilde{u}) = (\tilde{u}, \pi_{\tilde{u}}(F)) \leq (\tilde{u}, \pi_{\tilde{u}}(G)) = \sigma_G(\tilde{u})$ for all $\tilde{u} \in \beta I$, i.e. $\sigma_F \leq \sigma_G$. Conversely,

Let $F \not\subseteq G$. Then we find an ultrafilter $\tilde{u} \in \beta I$ satisfying $\pi_{\tilde{u}}(F) \not\subseteq \pi_{\tilde{u}}(G)$ in the following manner: Choose an $a \in F \setminus G$. Then for every $b \in G$ the set $M_b := \{i \in I\,;\ a(i) \not\geq b(i)\}$ is not empty, as otherwise we would have $a \geq b \in G$. The system $\mathfrak{M} := \{M_b\,;\ b \in G\}$ has the finite intersection property. Indeed, $\emptyset = M_{b_1} \cap \dots \cap M_{b_n}$ would imply $a \geq b_1 \wedge \dots \wedge b_n \in G$. Let $\tilde{u}$ be an ultrafilter containing $\mathfrak{M}$. Assume that $\pi_{\tilde{u}}(a) \in \pi_{\tilde{u}}(G)$. Then there is an $b \in G$ such that $\pi_{\tilde{u}}(a) = \pi_{\tilde{u}}(b)$, i.e. $Eq(a,b) = \{i\,;\ a(i) = b(i)\} \in \tilde{u}$. But then the empty set $\emptyset = M_b \cap Eq(a,b)$ is contained in $\tilde{u}$, a contradiction. Hence we have $\pi_{\tilde{u}}(a) \in \pi_{\tilde{u}}(F) \setminus \pi_{\tilde{u}}(G)$.
Last, let $\sigma_a = \sigma_{\downarrow a}$ for an $a \in \prod L_i$. Then $\{u\,;\ u \gg \sigma_a p(u)\}$ is open, what has been shown in example 2.4 (ii)

2.13) <u>Corollary</u>: Let L_i, $i \in I$ be lattices with 0 and 1 and let $p: \mathcal{E} \to \beta I$ be the bundle associated with the direct product of the L_i^{op}. Then for every $\tilde{u} \in \beta I$ the stalk $p^{-1}(u)$ is equal to $\{\tilde{u}\} \times \widehat{L_{\tilde{u}}^{op}} = \{\tilde{u}\} \times \mathcal{J}(L_{\tilde{u}})$, hence isomorphic to the ideal lattice $\mathcal{J}(L_{\tilde{u}})$ of $L_{\tilde{u}}$, and ideal lattice of $\prod L_i$ is isomorphic to $LC(p)$. The isomorphism $\sigma\_ : \mathcal{J}(\prod L_i) \to LC(p)$ is given by $J \mapsto \sigma_J$, where $\sigma_J(\tilde{u}) = (\tilde{u}, \pi_{\tilde{u}}(J))$ for all $\tilde{u} \in \beta I$. Moreover, if $J = \downarrow a$ is a principal ideal generated by $a \in \prod L_i$, then $\sigma_{\downarrow a} = \sigma_a$ is a continuous section such that $\{u\,;\ u \gg \sigma_a p(u)\}$ is open. Further, a selection σ is continuous if and only if $\sigma = \sigma_J$, where J is a product of ideals $J_i \in \mathcal{J}(L_i)$. Hence $T(p) \cong \prod \mathcal{J}(L_i)$.

The following remark we shall need in section 5:

2.14) <u>Remark</u> If $p: \mathcal{E} \to \beta I$ is the bundle associated with the direct product of the L_i^{op}, if $\tilde{u}$ is an ultrafilter on I and if $J \in \mathcal{J}(L_{\tilde{u}})$ is an ideal of $L_{\tilde{u}}$, then

$$\eta_{(\tilde{u},J)} = \sigma_{\pi_{\tilde{u}}^{-1}(J)}.$$

<u>Proof</u>: Clearly, $\sigma_{\pi^{-1}_{(\tilde{u})}(J)}(\tilde{u}) = (\tilde{u}, J) = \eta_{(\tilde{u},J)}(\tilde{u})$. If w is an ultrafilter different from $\tilde{u}$, choose an $M \in w \setminus \tilde{u}$ and define $a \in \prod L_i$ by $a(i) = 1$ for $i \in M$ and $a(i) = 0$ for $i \in I \setminus M$. Then $I \setminus M \in \tilde{u}$, hence $\pi_{\tilde{u}}(a) = \pi_{\tilde{u}}(0) = 0 \in J$ and $\pi_w(a) = \pi_w(1) = 1$. But this implies $a \in \pi_{\tilde{u}}^{-1}(J)$ and therefore $1 = \pi_w(a) \in \pi_w(\pi_{\tilde{u}}^{-1}(J))$. But this means exactly $\sigma_{\pi_{\tilde{u}}^{-1}(J)}(w) = (w, L_w) = \eta_{(\tilde{u},J)}(w)$.

3. Quotients of Prebundles

3.1) Definition: Let $p: \mathcal{E} \to X$ be a prebundle of continuous lattices and $\Theta \subseteq \mathcal{E} \times \mathcal{E}$ an equivalence relation. Then Θ is called a prebundle congruence provided that

(i) Θ is closed in $\mathcal{E} \times \mathcal{E}$

(ii) $\Theta \subseteq \ker p$

(iii) $\Theta_x := \Theta \cap (p^{-1}(x) \times p^{-1}(x))$ is an $\wedge$-congruence

If we start with a prebundle $p: \mathcal{E} \to X$ and a prebundle congruence Θ, we can construct a new prebundle in the following way:
Let $\mathcal{E}/\Theta$ be the quotient space of $\mathcal{E}$ modulo Θ, $\pi_\Theta: \mathcal{E} \to \mathcal{E}/\Theta$ the quotient map and $p_\Theta: \mathcal{E}/\Theta \to X$ the unique continuous map such that

$$\begin{tikzcd} \mathcal{E} \arrow[d, "p"'] \arrow[dr, "\pi_\Theta"] & \\ X & \mathcal{E}/\Theta \arrow[l, "p_\Theta"] \end{tikzcd}$$

commutes. Then the following proposition holds:

3.2) Proposition: $p_\Theta: \mathcal{E}/\Theta \to X$ is a prebundle of continuous lattices.

Proof. Clearly, $\mathcal{E}/\Theta$ and X are compact spaces and p_Θ is a surjective continuous map. Further $p_\Theta^{-1}(x) = p^{-1}(x)/\Theta \cong p^{-1}(x)/\Theta_x$ is a continuous lattice because Θ_x is a closed $\wedge$-congruence on $p^{-1}(x)$. Finally, $\wedge' : \operatorname{dom} \wedge' = \{(u,v) \in \mathcal{E}/\Theta \times \mathcal{E}/\Theta \,;\, p_\Theta(u) = p_\Theta(v)\} \to \mathcal{E}/\Theta$ is continuous. Indeed, the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi_\Theta"] & \operatorname{dom} \wedge \arrow[l, "\wedge"'] \arrow[d, "(\pi_\Theta \times \pi_\Theta)/\operatorname{dom} \wedge"] \\ \mathcal{E}/\Theta & \operatorname{dom} \wedge' \arrow[l, "\wedge'"] \end{tikzcd}$$

is commutative and $(\pi_\Theta \times \pi_\Theta)/\operatorname{dom} \wedge$ is a quotient map. Hence $\wedge'$ is continuous iff $\wedge' \circ (\pi_\Theta \times \pi_\Theta)/\operatorname{dom} \wedge$ is. But the latter is equal to $\pi_\Theta \circ \wedge$.

3.3) Notation: $p_\Theta: \mathcal{E}/\Theta \to X$ is called the quotient prebundle of $p: \mathcal{E} \to X$ modulo Θ.

3.4) Proposition: Let $p : \mathcal{E} \to X$ be a prebundle and let Θ be a prebundle congruence. Then for every $\sigma \in LC(p)$ the section $\overline{\pi}_\Theta(\sigma) := \pi_\Theta \circ \sigma$ is contained in $LC(p_\Theta)$.

Proof. Let $(x_i)_{i\in I}$ be a convergent net of X and u' be a cluster point of $(\pi_\Theta \circ \sigma(x_i))_{i\in I}$. Then there exists a cluster point u of $(\sigma(x_i))_{i\in I}$ such that $\pi_\Theta(u) = u'$. But the lower semicontinuity of σ implies $u \geq \sigma(\lim x_i)$ and hence $u' = \pi_\Theta(u) \geq \pi_\Theta \circ \sigma(\lim x_i)$.

3.5) Remark The map $\overline{\pi}_\Theta : LC(p) \to LC(p_\Theta)$ is order preserving.

3.6) Proposition Let $\sigma \in LC(p_\Theta)$. Then the map $d_\Theta(\sigma) : X \to \mathcal{E}$ defined by $d_\Theta(\sigma)(x) = \inf p_\Theta^{-1}(\sigma(x))$ is lower semicontinuous. Moreover, $\overline{\pi}_\Theta \circ d_\Theta = id_{LC(p_\Theta)}$ and $d_\Theta(\sigma) \leq \tau$ iff $\sigma \leq \overline{\pi}_\Theta(\tau)$ for all $\tau \in LC(p)$. Hence, d_Θ is the right adjoint of $\overline{\pi}_\Theta$ and $\overline{\pi}_\Theta$ is onto.

Proof. $G(d_\Theta(\sigma)) = \pi_\Theta^{-1}(G(\sigma))$ and therefore the continuity of π_Θ implies the lower semicontinuity of $d_\Theta(\sigma)$.

3.7) Proposition The map $\overline{\pi}_\Theta : LC(p) \to LC(p_\Theta)$ preserves arbitrary infima and updirected suprema.

Proof. Every left adjoint preserves infima and $\overline{\pi}_\Theta$ is the left adjoint of d_Θ. That $\overline{\pi}_\Theta$ preserves updirected suprema follows from the fact that for every $x \in X$ the restriction $\pi_{\Theta / p^{-1}(x)} : p^{-1}(x) \to p_\Theta^{-1}(x)$ preserves updirected suprema and that suprema are calculated pointwise.

3.8) Corollary: If $LC(p)$ is a continuous lattice, then so is $LC(p_\Theta)$.

3.9) Corollary: If $p : \mathcal{E} \to X$ is a quasibundle, then $LC(p_\Theta)$ is a continuous lattice

3.10) Proposition: Let $p : \mathcal{E} \to X$ be a quasibundle and Ψ be a closed $\wedge$-congruence on $LC(p)$. Then $\Theta_\Psi := \{(u,v) \in \mathcal{E} \times \mathcal{E} \,;\, p(u) = p(v) \text{ and } (\eta_u, \eta_v) \in \Psi\}$ is a prebundle congruence. Moreover, there exists an unique CL-morphism φ such that

$$\begin{tikzcd}
LC(p) \arrow[d, "\pi_\Psi"'] \arrow[dr, "\overline{\pi}_{\Theta_\Psi}"] & \\
LC(p)/_\Psi & LC(p_{\Theta_\Psi}) \arrow[l, "\varphi"]
\end{tikzcd}$$

commutes

Proof: $\Theta_\Psi = (\bar{\eta}_- \times \eta_-)^{-1}(\Psi) \cap \ker p$, hence Θ_Ψ is a closed equivalence relation contained in $\ker p$. Clearly, $\Theta_\Psi \cap (p^{-1}(x) \times p^{-1}(x))$ is a $\wedge$-congruence, hence Θ_Ψ is a prebundle congruence. Finally, an easy computation shows $\ker(\pi_{\Theta_\Psi}) \subseteq \Psi$.

3.11) Definition: The prebundle congruence Θ_Ψ is called the prebundle congruence induced by Ψ

B) Representation of Coproducts by Lower semicontinuous Sections in Bundles

4. Ultraproducts in SRIP

For definitions we refer to [2].

Let $(S_i, \sqsubset_i)$, $i \in I$, be SRIP-objects. An easy calculation shows that $(\prod S_i, \sqsubset)$ is again a SRIP-object, where $a \sqsubset b$ if and only if $a(i) \sqsubset_i b(i)$ for all $i \in I$. – Now let $\tilde{u}$ be an ultrafilter on I. Then we may define the model theoretical ultraproduct $(S_{\tilde{u}}, \sqsubset_{\tilde{u}})$ of the $(S_i, \sqsubset_i)$. Clearly, SRIP-objects being defined in a first order language, $(S_{\tilde{u}}; \sqsubset_{\tilde{u}})$ is again a SRIP-object.

If $(S, \sqsubset)$ is a SRIP-object, denote by $\mathcal{I}(S)$ the lattice of all ideals of S and by $\mathcal{I}_{\sqsubset}(S)$ the (continuous) lattice of all $\sqsubset$-ideals. Further, let $c: \mathcal{I}(S) \to \mathcal{I}_{\sqsubset}(S)$ the mapping, which assigns to every ideal $I \in \mathcal{I}(S)$ the largest $\sqsubset$-ideal $c(I)$ contained in I. Then c is an CL-morphism. Recall that for $I, J \in \mathcal{I}_{\sqsubset}(S)$ we have $I \ll J$ if and only if $I \subseteq c(\downarrow a)$ for some $a \in J$.

4.1) Proposition: Let $(S_i, \sqsubset_i)$, $i \in I$, be SRIP-objects and $\tilde{u}$ an ultrafilter on I. Further, let $\pi_{\tilde{u}}: \prod S_i \to S_{\tilde{u}}$ the quotient map. Then

(i) $\pi_{\tilde{u}}$ is a SRIP-morphism

(ii) $\pi_{\tilde{u}}$ maps $\sqsubset$-ideals of $\prod S_i$ onto $\sqsubset_{\tilde{u}}$-ideals of $S_{\tilde{u}}$

(iii) $\pi_{\tilde{u}}(c(\downarrow a)) = c(\downarrow \pi_{\tilde{u}}(a))$.

(iv) $\pi_{\tilde{u}}$ is right adjoint to $\mathcal{I}_{\sqsubset}(\pi_{\tilde{u}}): \mathcal{I}_{\sqsubset}(S_{\tilde{u}}) \to \mathcal{I}_{\sqsubset}(\prod S_i)$, $J \mapsto c(\pi_{\tilde{u}}^{-1}(J))$.

Proof. (i) is clear by the definition of the ultraproduct.

(ii) First, let $J \in \mathcal{I}(\prod S_i)$ be an ideal. Clearly, $\pi_{\tilde{u}}(J)$ is closed under finite sup's. So let $a \in J$, $b \in \prod S_i$ such that $\pi_{\tilde{u}}(b) \leq \pi_{\tilde{u}}(a)$. We want to show that $\pi_{\tilde{u}}(b) \in \pi_{\tilde{u}}(J)$. But $\pi_{\tilde{u}}(b) \leq \pi_{\tilde{u}}(a)$ implies $M := \{i \in I; b(i) \leq a(i)\} \in \tilde{u}$. Define a new element $\bar{b} \in \prod S_i$ by $\bar{b}(i) = b(i)$ for $i \in M$ and $\bar{b}(i) = 0$ for $i \in I \setminus M$. Then $\bar{b} \leq a$, hence $\bar{b} \in J$. Moreover $\pi_{\tilde{u}}(b) = \pi_{\tilde{u}}(\bar{b}) \in \pi_{\tilde{u}}(J)$. – Now let $J \in \mathcal{I}_{\sqsubset}(\prod S_i)$ be an $\sqsubset$-ideal and $\pi_{\tilde{u}}(a) \in \pi_{\tilde{u}}(J)$. W.l.o.g. we may assume that $a \in J$. But then there is an $b \in J$ satisfying $a \sqsubset b$. Clearly, $\pi_{\tilde{u}}(b) \in \pi_{\tilde{u}}(J)$ and $\pi_{\tilde{u}}(a) \sqsubset_{\tilde{u}} \pi_{\tilde{u}}(b)$. Hence $\pi_{\tilde{u}}(J)$ is an $\sqsubset_{\tilde{u}}$-ideal.

(iii) Clearly, $\pi_{\tilde u}(c(\downarrow a)) \subseteq c(\downarrow \pi_{\tilde u}(a))$ by (ii). Conversely, let $\pi_{\tilde u}(b) \sqsubset \pi_{\tilde u}(a)$. Then the set $M := \{i \in I \,;\, b(i) \sqsubset_i a(i)\}$ is contained in $\tilde u$. Define $\bar b \in \Pi S_i$ by $\bar b(i) = b(i)$ for $i \in M$ and $\bar b(i) = 0$ for $i \in I \setminus M$. Then $\bar b(i) \sqsubset a(i)$ for all $i \in I$, hence $\bar b \sqsubset a$. But this means $\bar b \in c(\downarrow a)$. Therefore $\pi_{\tilde u}(b) = \pi_{\tilde u}(\bar b) \in \pi_{\tilde u}(c(\downarrow a))$.

(iv) Let $\pi_{\tilde u}(J) \subseteq J'$ for $J \in J_{\sqsubset}(\Pi S_i)$ and $J' \in J_{\sqsubset}(S_{\tilde u})$. Then $J \subseteq \pi_{\tilde u}^{-1}(J')$. But J is an $\sqsubset$-ideal, hence $J \subseteq c(\pi_{\tilde u}^{-1}(J'))$. Conversely, if $J \subseteq c(\pi_{\tilde u}^{-1}(J'))$ hold, then $J \subseteq \pi_{\tilde u}^{-1}(J')$ holds, too. But this implies $\pi_{\tilde u}(J) \subseteq J'$.

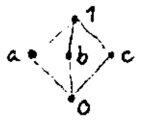
<u>For the rest of this section we assume that all the</u> S_i <u>are</u> <u>CL-objects and that</u> $\sqsubset_i$ <u>is the way below relation on</u> S_i. Especially, $S_{\tilde u}$ is always a lattice.

4.2) <u>Proposition</u> (i) If $a, b \in S_{\tilde u}$ and if $a \not\leq b$, then there exists an $c \in S_{\tilde u}$ satisfying $c \sqsubset_{\tilde u} a$ and $c \not\leq b$.

(ii) If $x \sqsubset_{\tilde u} a \vee b$, then there exist $y \sqsubset_{\tilde u} a$ and $z \sqsubset_{\tilde u} b$ such that $x \sqsubset_{\tilde u} y \vee z$.

Proof. (i) and (ii) are satisfied in every continuous lattice and hence by Łoš's theorem in $(S_{\tilde u}, \sqsubset_{\tilde u})$. (For Łoš's theorem see Bell - Slomson: Models and Ultraproducts, North-Holland Publishing Company <u>1971</u>)

4.3) <u>Remark</u> In general, (i) and (ii) are not true in <u>SRIP</u> (or even in <u>CSRIP</u>), as the counter example



with $0 \sqsubset a$, $0 \sqsubset b$, $0 \sqsubset c$, $0 \sqsubset 1$, $0 \sqsubset 0$, $a \sqsubset 1$, $a \sqsubset a$ shows

4.4) <u>Proposition</u> The mapping $a \mapsto c(\downarrow a) : S_{\tilde u} \to J_{\sqsubset}(S_{\tilde u})$ is one-to-one, preserves finite infima and suprema and satisfies $a \sqsubset_{\tilde u} b$ if and only if $c(\downarrow a) \ll c(\downarrow b)$. Especially, $a \mapsto c(\downarrow a)$ is an embedding of <u>SRIP</u>-objects onto a dense substructure of $J_{\sqsubset}(S_{\tilde u})$.

Proof. Clearly, $a \mapsto c(\downarrow a)$ is one-to-one by 4.2 (i). Moreover, $c(\downarrow a) \wedge c(\downarrow b) = c(c(\downarrow a) \cap c(\downarrow b)) =$
$= \{c \in S_{\tilde u} \,;\, \exists c' \in c(\downarrow a) \cap c(\downarrow b) \text{ s.t. } c \sqsubset_{\tilde u} c'\} = \{c \in S_{\tilde u} \,;\, \exists c' \in S_{\tilde u} : c \sqsubset_{\tilde u} c' \sqsubset_{\tilde u} a, b\} \subseteq \{c \in S_{\tilde u} ;\, \exists c' : c \sqsubset_{\tilde u} c' \leq a \wedge b\} =$
$= \{c \in S_{\tilde u} \,;\, c \sqsubset a \wedge b\} = c(\downarrow a \wedge b) \subseteq c(\downarrow a) \wedge c(\downarrow b)$. Hence $a \mapsto c(\downarrow a)$ preserves finite infima.
Further, $a \mapsto c(\downarrow a)$ preserves finite suprema: First, note that 4.2 (ii) implies $\downarrow \{x \vee y \,;\, x \sqsubset_{\tilde u} a,\, y \sqsubset_{\tilde u} b\} =$

$= \downarrow\{z\ ;\ z \sqsubset_{\tilde u} a \vee b\}$. Therefore the equality $c(\downarrow a) \vee c(\downarrow b) = \downarrow\{x \vee y\ ;\ x \sqsubset_{\tilde u} a,\ y \sqsubset_{\tilde u} b\} =$

$= \downarrow\{z\ ;\ z \sqsubset_{\tilde u} a \vee b\} = c(\downarrow a \vee b)$ holds.

Lastly, let $a \sqsubset_{\tilde u} b$. Then the interpolation property yields an element $c \in S_{\tilde u}$ satisfying $a \sqsubset_{\tilde u} c \sqsubset_{\tilde u} b$. This implies $c(\downarrow a) \leq \downarrow c \leq c(\downarrow b)$, i.e. $c(\downarrow a) \ll c(\downarrow b)$. Conversely, let $c(\downarrow a) \ll c(\downarrow b)$. Then there exists an $c \in c(\downarrow b)$ such that $c(\downarrow a) \leq \downarrow c$. Assume, if possible, that $a \not\leq c$. Then by 4.2 (i) we may find an $x \in S_{\tilde u}$ such that $x \sqsubset_{\tilde u} a$ and $x \not\leq c$, a contradiction to $c(\downarrow a) \leq \downarrow c$. Hence $a \leq c \sqsubset_{\tilde u} b$.

If all the S_i are equal to S, then S may be embedded in the ultraproduct $S_{\tilde u}$ via the diagonal. Let $i_{\tilde u}: S \to S_{\tilde u}$ be this embedding. Clearly, $i_{\tilde u}$ is a lattice embedding satisfying $x \ll y$ if and only if $i_{\tilde u}(x) \sqsubset_{\tilde u} i_{\tilde u}(y)$. Especially, $i_{\tilde u}$ is a SRIP-morphism and hence $x \mapsto c(\downarrow i_{\tilde u}(x)) : S \to J_{\sqsubset}(S_{\tilde u})$ is a SRIP-morphism, too. These observations together with proposition 4.4 prove the first part of

4.5) Proposition If all the S_i are equal to S, then the SRIP-morphism $d_{\tilde u} : x \mapsto c(\downarrow i_{\tilde u}(x))\ ;\ S \to J_{\sqsubset}(S_{\tilde u})$ is a lattice embedding satisfying $x \ll y$ iff $c(\downarrow i_{\tilde u}(x)) \ll c(\downarrow i_{\tilde u}(y))$. Moreover, $J \mapsto \sup i_{\tilde u}^{-1}(J) : J_{\sqsubset}(S_{\tilde u}) \to S$ is a surjective CL-morphism.

Proof. $J \mapsto \sup i_{\tilde u}^{-1}(J)$ is surjective, as $x = \sup i_{\tilde u}^{-1}(d(x))$. Moreover, $J \mapsto \downarrow i_{\tilde u}^{-1}(J) = \{y \in S\ ;\ \exists a \in J : y \ll i_{\tilde u}(a)\} : J_{\sqsubset}(S_{\tilde u}) \to J_{\ll}(S)$ is a CL-morphism by [, (3.14)]. Because $J' \mapsto \sup J' : J_{\ll}(S) \to S$ is an isomorphism and because $\sup \downarrow i_{\tilde u}^{-1}(J) = \sup i_{\tilde u}^{-1}(J)$ holds, the mapping $J \mapsto \sup i_{\tilde u}^{-1}(J)$ is a CL morphism.

4.6) Remark: The right adjoint of $J \mapsto \sup i_{\tilde u}^{-1}(J)$ is the mapping $x \mapsto \{a \in S_{\tilde u}\ ;\ \exists y \in S : y \ll x \text{ and } a \leq i_{\tilde u}(y)\}$. This mapping is in general different from $d_{\tilde u}$. Indeed, let $S = [0,1]$ be the unit intervall an $\tilde u$ be a free ultrafilter on $\mathbb{N}$. Let $\pi_{\tilde u} : S^{\mathbb{N}} \to S$ the projection. Define an element $a \in S^{\mathbb{N}}$ by $a(n) = 1 - 1/n$. Then we have $a \sqsubset 1$, hence $\pi_{\tilde u}(a) \in \pi_{\tilde u}(c\downarrow 1) = c(\downarrow \pi_{\tilde u}(1)) = c(\downarrow i_{\tilde u}(1)) = d_{\tilde u}(1)$. But there exists no $c \in [0,1]$ satisfying $c < 1$ and $\pi_{\tilde u}(a) \leq i_{\tilde u}(c)$, because $\pi_{\tilde u}(a) \leq i_{\tilde u}(c)$ implies $\{n \in \mathbb{N}\ ;\ a(n) \leq c\} \in \tilde u$ which is contradictory to $\lim_{\tilde u} a_n = 1$. Hence $\pi_{\tilde u}(a) \notin \{b \in S_{\tilde u}\ ;\ \exists y \in S : y \ll 1 \text{ and } b \leq i(y)\}$.

A similar argument shows that $d_{\tilde u}$ does not preserve arbitrary infima.

4.7) Questions: Is $d_{\tilde u}(S)$ discrete in the Lawson topology of $J_{\sqsubset}(S_{\tilde u})$? – Is $J_{\sqsubset}(S_{\tilde u})$ generated by $d_{\tilde u}(S)$?

5. The Representation Theorem

In this section we shall prove the following theorem by several lemmas:

5.1) Theorem Let S_i, $i\in I$, be continuous lattices. Then there exists a bundle $p: \mathcal{E}\to\beta I$ of continuous lattices such that $\coprod S_i \cong LC(p)$.

More precisely, you can choose $\mathcal{E} := \bigcup_{\tilde{\alpha}\in\beta I} (\tilde{\alpha}, \mathrm{J}_c(S_{\tilde{\alpha}}))$ and $p: (\tilde{\alpha}, J)\mapsto \tilde{\alpha} : \mathcal{E}\to\beta I$.

The topology on $\mathcal{E}$ may be constructed in the following manner: let $p': \mathcal{E}'\to\beta I$ be the bundle associated with the direct product of the S_i^{op}. Then $\mathcal{E}' = \bigcup_{\tilde{\alpha}\in\beta I} (\tilde{\alpha}, \mathrm{J}(S_{\tilde{\alpha}}))$ is a compact space. Give $\mathcal{E}$ the quotient topology induced by the map $\pi_c : (\tilde{\alpha}, J)\mapsto(\tilde{\alpha}, c(J)) : \mathcal{E}'\to\mathcal{E}$.

Furthermore, $\Gamma(p)$ is isomorphic to $\prod S_i$ and for every $\sigma\in\Gamma(p)$, the set $\{u\in\mathcal{E};\ u \gg \sigma p(u)\}$ is open.

The coprojections $\eta_i : S_i \longrightarrow LC(p)$ are given by $a \longmapsto \eta_{\bar{a}}$, where $\bar{a} = (i, \downarrow a)$.

5.2) Remark: Clearly, for every $i\in I\subseteq\beta I$, the stalk of p over i is isomorphic with S_i.

Now we start the proof of 5.1: Let $\sigma' : \mathrm{J}(\prod S_i)\to LC(p')$ be the canonical isomorphism. Then

5.3) Lemma $\psi : LC(p) \xrightarrow{\sigma'^{-1}} \mathrm{J}(\prod S_i) \xrightarrow{c} \mathrm{J}_c(\prod S_i) \cong \coprod S_i$ is a continuous surjective homomorphism of $LC(p)$ onto $\coprod S_i$ (see [3 ∃]).

5.4) Lemma: $\ker \pi_c$ is the prebundle congruence induced by $\ker\psi$, hence $p: \mathcal{E}\to\beta I$ is a prebundle.

Proof. We have to show that $\pi_c((\tilde{\alpha}, J_1)) = \pi_c((\tilde{\alpha}, J_2))$ holds if and only if $\psi(\eta_{(\tilde{\alpha},J_1)}) = \psi(\eta_{(\tilde{\alpha},J_2)})$ holds. First, note that $\eta_{(\tilde{\alpha},J_1)} = \sigma'_{\pi_{\tilde{\alpha}}^{-1}(J)}$ by 2.14. Hence $\psi(\eta_{(\tilde{\alpha},J_1)}) = \psi(\eta_{\tilde{\alpha},J_2})$ is equivalent to $c(\pi_{\tilde{\alpha}}^{-1}(J_1)) = c(\pi_{\tilde{\alpha}}^{-1}(J_2))$. So we have to show that $c(\pi_{\tilde{\alpha}}^{-1}(J_1)) = c(\pi_{\tilde{\alpha}}^{-1}(J_2))$ is equivalent

~~to $c(J_1)=c(J_2)$. First, let $c(J_1)=c(J_2)$. Then, by 4.1.(ii) we have $\pi_{\tilde{\alpha}}(c(\pi_{\tilde{\alpha}}^{-1}(J_1)))\subseteq c(J_1)=c(J_2)$~~

Further, 4.1 (iv) implies $c(\pi_{\tilde{\alpha}}^{-1}(J_1)) \subseteq c(\pi_{\tilde{\alpha}}^{-1}(c(J_2))) \subseteq c(\pi_{\tilde{\alpha}}^{-1}(J_2))$. – In the same way we get $c(\pi_{\tilde{\alpha}}^{-1}(J_2)) \subseteq c(\pi_{\tilde{\alpha}}^{-1}(J_1))$. Conversely, let $c(\pi_{\tilde{\alpha}}^{-1}(J_1)) = c(\pi_{\tilde{\alpha}}^{-1}(J_2))$. If we are able to prove

So let $\pi_{\tilde u}(a) \in c(J_\cdot)$. Then there exists an $b \in \pi_{\tilde u}^{-1}(J_\cdot)$ such that $\pi_{\tilde u}(a) \sqsubset_\infty \pi_{\tilde u}(b)$. Therefore the set $M := \{i \in I \,;\, a(i) \ll b(i)\}$ is contained in $\tilde u$. Define an element $\bar a \in \prod S_i$ by $\bar a(i) = a(i)$ for $i \in M$ and $\bar a(i) = 0$ for $i \in I \setminus M$. Then $\bar a \sqsubset b$ and $\pi_{\tilde u}(\bar a) = \pi_{\tilde u}(a)$. But $\bar a$ is an element of $c(\pi_{\tilde u}^{-1}(J_\cdot))$. Thus we have shown that $\pi_a(a) = \pi_{\tilde u}(\bar a) \in \pi_{\tilde u}(c(\pi_{\tilde u}^{-1}(J_\cdot)))$. This concludes the proof.

5.5) <u>Lemma</u>: Every lower semicontinuous selection of p is a updirected supremum of continuous one's

Proof. The claim is true for the bundle associate with the direct product of the L_i^{op} by 2.9. As $LC(p)$ is a continuous lattice (see 3.9) and as $\overline{\pi}_c : LC(p') \twoheadrightarrow LC(p) : \sigma \mapsto \pi_c \circ \sigma$ is a surjective <u>CL</u>-moplism mapping $\Gamma(p')$ into $\Gamma(p)$ (see 3.7), the claim is true for $p : \mathcal{E} \to \beta I$, too.

5.6) <u>Lemma</u>: $\sigma : \beta I \to \mathcal{E}$ is continuous iff there exists an $a \in \prod S_i$ such that $\sigma = \sigma_a$, where $\sigma_a(\tilde u) = (\tilde u, \pi_{\tilde u}(c \downarrow a))$

Proof. Clearly, the continuity of $\overline{\pi}_c$ and Corollary 2.13 implies the continuity of σ_a. Conversely, let $\sigma : \beta I \to \mathcal{E}$ be continuous. Define $a \in \prod S_i$ by $a(i) = \sup \sigma(i)$ $(\sigma(i) \in \mathcal{J}_\sqsubset(S_i)\,!!)$

Then we have $\sigma_a(i) = \pi_i(c \downarrow a) \overset{4.2(ii)}{=} c(\downarrow \pi_i(a)) = c(\downarrow a(i)) = c(\downarrow \sup \sigma(i)) = \{a \in S_i \,;\, a \ll \sup \sigma(i)\} = \sigma(i)$.

Hence σ_a and σ agree on the dense set I and therefore they are equal.

7) <u>Corollary</u> $\Gamma(p) \cong \prod S_i$

8) <u>Lemma</u> Let $d : LC(p) \to LC(p')$ be the right adjoint of $\overline{\pi}_c : LC(p') \twoheadrightarrow LC(p)$. Then for every continuous section $\sigma : \beta I \to \mathcal{E}$ the selection $d(\sigma)$ is continuous and $\{u \in \mathcal{E}' \,;\, u \gg d(\sigma)(p'(u))\}$ is open.

Proof: $d(\sigma)(\tilde u) = \sigma(\tilde u)$ for all $\tilde u \in \beta I$ (Recall that $\mathcal{J}_\sqsubset(S_{\tilde u}) \subseteq \mathcal{J}(S_{\tilde u})$). If σ is continuous, then there is an $a \in \prod S_i$ satisfying $\sigma = \sigma_a$. Further, by definition of $\sqsubset$ on $\prod S_i$ we have

$c(\downarrow a) = \{ b \in \prod S_i \; ; \; b(i) \ll a(i) \text{ for all } i \in I \}$. Hence $c(\downarrow a)$ is the product of the ideals $\downarrow a(i) = \{ b \in S_i \; ; \; b \ll a(i) \} \in \mathcal{J}(S_i)$. Therefore $\alpha(\sigma_a) = \sigma'_{c(\downarrow a)}$ is continuous by 2.13. Moreover, $\{ u \in \mathcal{E}' \; ; \; u \gg \alpha(\sigma_a)(p'(u)) \}$ is open. Indeed, let $(\tilde{u}, J) \gg (\tilde{u}, \pi_{\tilde{u}}(c \downarrow a)) = (\tilde{u}, c(\downarrow \pi_{\tilde{u}}(a))$. Then there exists an $x \in J$ such that $\downarrow x \supseteq c(\downarrow \pi_{\tilde{u}}(a))$. But this implies $c(\downarrow x) \supseteq c(\downarrow \pi_{\tilde{u}}(a))$ and hence $x \geq \pi_{\tilde{u}}(a)$ by 4.4. This means $(\tilde{u}, J) \gg (\tilde{u}, \downarrow \pi_{\tilde{u}}(a)) = \sigma'_a(\tilde{u})$. (For the definition of σ'_a see 2.13). Thus, we have just shown that $\{ u \in \mathcal{E}' \; ; \; u \gg \alpha(\sigma_a)(p'(u)) \} = \{ u \in \mathcal{E}' \; ; \; u \gg \sigma'_a(p'(u)) \}$. But the latter is open by 2.13.

5.9) <u>Lemma</u>: $\{ u \in \mathcal{E} \; ; \; u \gg \sigma p(u) \}$ is open for every continuous section $\sigma : \mathcal{E} \to \beta I$.

Proof. We shall prove that $\pi_c^{-1} \{ u \in \mathcal{E} \; ; \; u \gg \sigma p(u) \}$ is open. First, let $\sigma = \sigma_a$ for some $a \in \prod S_i$. If $\pi_c(\tilde{u}, J)$ is way above $(\tilde{u}, c(\downarrow \pi_{\tilde{u}}(a))$, then there exists an $y \in c(J)$ such that $\downarrow y \supseteq c(\downarrow \pi_{\tilde{u}}(a))$. Proposition 4.2 (i) implies $y \geq \pi_{\tilde{u}}(a)$. Choose an $b \in \prod S_i$ which satisfies $\pi_{\tilde{u}}(b) \in J$ and $\pi_{\tilde{u}}(b)_{\tilde{u}} = y \geq \pi_{\tilde{u}}(a)$. Then the set $M := \{ i \in I \; ; \; b(i) \gg a(i) \}$ is contained in $\tilde{u}$ and hence $\mathcal{U} := \{ w \in \beta I \; ; \; M \in w \}$ is an open neighborhood of $\tilde{u}$. Then the set $\mathcal{O} := p^{-1}(\mathcal{U}) \cap \{ u \in \mathcal{E}' \; ; \; u \gg \sigma'_b p'(u) \}$ is an open neighborhood of $(\tilde{u}, J)$ contained in $\pi_c^{-1} \{ u \in \mathcal{E} \; ; \; u \gg \sigma p(u) \}$. Indeed, $\mathcal{O}$ is open and contains $(\tilde{u}, J)$, because $(\tilde{u}, J) \gg (\tilde{u}, \downarrow \pi_{\tilde{u}}(b)) = \sigma_b(\tilde{u})$. Moreover, if $(w, J') \in \mathcal{O}$, then M is contained in w and the equation $\pi_w(b) \in J'$ holds. But this implies $\pi_w(a) \sqsubset_w \pi_w(b)$ and hence $c(\downarrow \pi_w(a)) \ll J'$. But this is exactly to say $\pi_c(w, J') \in$ $\in \{ u \in \mathcal{E} \; ; \; u \gg \sigma_a p(u) \}$.

5.10) <u>Corollary</u>: $p : \mathcal{E} \to \beta I$ is a bundle

Proof. Easy calculation using lemma 5.5 and lemma 5.9.

Now let $\varphi : LC(p) \longrightarrow \mathcal{J}_c(\prod S_i) \cong \coprod S_i$ the unique <u>CL</u>-morphism and let $pr_i : S_i \to \mathcal{J}_c(\prod S_i)$ be the coprojections. Recall that $pr_i(s) = \{ a \in \prod S_i \; ; \; a(i) \ll s \text{ and } a(j) \ll 1 \text{ for } i \neq j \} = c(\pi_i^{-1}(\downarrow s))$

5.11) <u>Lemma</u> The diagram

$$\begin{tikzcd}
\mathcal{J}(\pi S_i) \arrow[rr, "\sigma'_-"] \arrow[d, "c"] & & LC(p') \arrow[d, "\bar{\pi}_c"] \\
\mathcal{J}_c(\pi S_i) & & LC(p) \arrow[ll, two heads, "\varphi"] \\
& S_i \arrow[ul, "pr_i"] \arrow[ur, "\eta_-"'] &
\end{tikzcd}$$

commutes.

Proof. By definition, $\varphi \circ \bar{\pi}_c = c \circ \sigma_-'^{-1}$, hence the equation $c = \varphi \circ \bar{\pi}_c \circ \sigma_-'^{-1}$.

Further, for all $a \in S_i$ we have $\varphi(\eta_-(a)) = \varphi(\eta_{\tilde{a}}) = \varphi(\bar{\pi}_c(\eta'_a))$, where $\eta'_a : \beta I \to \mathcal{E}'$ is defined by $\eta'_a(i) = (i, \downarrow a)$ and $\eta'_a(\tilde{x}) = (\tilde{x}, S_{\tilde{x}})$ for all $\tilde{x} \neq i$. Hence we can write $\varphi(\eta_-(a)) = \varphi(\bar{\pi}_c(\eta'_a)) = \varphi \circ \bar{\pi}_c \circ \sigma'_-(\pi_i^{-1}(\downarrow a))$, because $\eta'_a = \sigma'_{\pi_i^{-1}(\downarrow a)}$ by 2.14. This implies $\varphi(\eta_-(a)) = c(\pi_i^{-1}(\downarrow a)) = pr_i(a)$.

5.12) <u>Corollary</u>: φ is an isomorphism.

Proof. $\eta_- : S_i \to LC(p)$ is a CL-morphism by 2.10 and $\bigcup_{i \in I} \eta_-(S_i)$ generates $LC(p)$ by 2.11. Hence there exists a unique surjective <u>CL</u>-morphism $\gamma : \mathcal{J}_c(\pi S_i) \twoheadrightarrow LC(p)$. A standard argument using 5.11 gives $\varphi \circ \gamma = id$, hence γ is injective. Therefore we have $\varphi = \gamma^{-1}$ is bijective.

The proof of the theorem follows from the corollaries 5.7, 5.10, 5.12 and from lemma 5.9.

.13) <u>Proposition</u>: $\bigcup(\eta_- S_i)$ is dense in the compact set $\eta(\mathcal{E}) \subseteq LC(p) \cong \coprod S_i$

Proof. Clearly, the closure of $\bigcup(\eta_- S_i)$ contains $\{\eta_{\sigma(\tilde{x})} ; \tilde{x} \in \beta I$ and $\sigma \in T(p)\}$. But $\{\sigma(\tilde{x}) ; \sigma \in T(p)\}$ is dense in $p^{-1}(\tilde{x})$.

.14) <u>Corollary</u> (HOFMANN, [5, Theorem 7]): $\coprod S_i \cong BF(p)^{op}$

6. More about the Stalks

Let S_i, $i \in I$, be continuous lattices and $p : \mathcal{E} \to \beta I$ be the bundle constructed in section 5 satisfying $\prod S_i \cong LC(p)$. Then for every $\tilde{u} \in \beta I$ the stalk $p^{-1}(u) \cong \mathcal{I}_{\sqsubset}(S_{\tilde{u}})$ is very large; indeed it contains the big and fat algebraic ultraproduct $S_{\tilde{u}}$ as a dense substructure (see proposition 4.4). In this section we shall prove that many first order properties, which hold in the S_i, are true in all stalks of $p : \mathcal{E} \to \beta I$.

The *language of lattice* $\mathcal{L}_L$ is defined to be the set of all first order formulas build from the quantifiers $\forall, \exists$, the connectives AND, OR and NOT, the lattice operations $\wedge$ and $\vee$ and the relation „$\leq$". By the *language of continuous lattice* $\mathcal{L}_{CL}$ I understand the first order language build from $\forall, \exists$, AND, OR, NOT, $\wedge, \vee, \leq$ and the slloop relation „$\sqsubset$", where in a continuous lattice the slloop relation always should be interpreted by the way below relation. Note that $\underline{CL} \subsetneq Md_{\mathcal{L}_{CL}}(Th_{\mathcal{L}_{CL}}(\underline{CL})) \subsetneq \underline{SRIP}$

Recall that a sentence is a formula without free variables. A sentence is said to be positive universal if it can be build from the operations and relations, the connectives AND and OR and the quantifier $\forall$. An existential sentence is a sentence of the form $(\exists x)(\exists y) \ldots \phi$, where ϕ is a formula not containing quantifiers.

Last, let $\tilde{u}$ be an ultrafilter on I. A sentence ϕ is said to be true in $\tilde{u}$-almost all the S_i, if $\{i \in I,\ S_i \models \phi\}$ is contained in $\tilde{u}$.

Let's first have a closer look on the lattice $\mathcal{I}_{\sqsubset}(S_{\tilde{u}})$.

6.1) Proposition: $\mathcal{I}_{\sqsubset}(S_{\tilde{u}})$ is a homomorphic image of a sublattice of an ultraproduct of the S_i, in symbols $\mathcal{I}_{\sqsubset}(S_{\tilde{u}}) \in HSP_U(\{S_i; i \in I\})$

Proof. K.A. BAKER and A.W. HALES [1] have shown that $\mathcal{I}(S_{\tilde{u}}) \in HSP_U(S_{\tilde{u}})$. As $c : \mathcal{I}(S_{\tilde{u}}) \twoheadrightarrow \mathcal{I}_{\sqsubset}(S_{\tilde{u}})$ is a lattice homomorphism and as $S_{\tilde{u}} \in P_U(\{S_i; i \in I\})$, we contain $\mathcal{I}_{\sqsubset}(S_{\tilde{u}}) \in HHSP_U P_U(\{S_i; i \in I\}) = HSP_U(\{S_i; i \in I\})$.

6.2) Proposition: Every existential sentence ϕ (in the language of continuous lattices), which holds in $\tilde{u}$-almost all of the S_i, is true in $\mathcal{I}_{\sqsubset}(S_{\tilde{u}})$.

Proof. If $\{i;\ S_i \models \phi\}$ is contained in $\tilde{u}$, then $S_{\tilde{u}} \models \phi$ by Łoś's theorem. But $S_{\tilde{u}}$ is a substructure of $\mathcal{I}_{\sqsubset}(S_{\tilde{u}})$ by proposition 4.4. Further, if an existential sentence is true in a substructure, then it is true in the larger structure. This prove 6.2.

6.3) Proposition (see BAKER, HALES): Let ϕ be a positive universal sentence in the language of lattices. If $\tilde{u}$-almost all of the S_i satisfy ϕ, then so does $\mathcal{I}_{\sqsubset}(S_{\tilde{u}})$.

P. 1.., Clearly, under the assumptions of 6.3 we have $S_{\tilde{u}} \models \phi$. This proves the theorem, as positive universal sentences are preserved by the operators H, S and P_U.

The following properties may be expressed by positive universal sentences (see K.A. BAKER: Equational axioms for classes of lattices, Bull. Amer. Math. Soc. 77 (1971), 97-102):

The lattice L satisfies the lattice equation $p=q$ (for instance $x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)$)

" " " is totally ordered

" " " has at most width n

" " " " " length n

" " " " " " breadth "

6.4) Corollary (.... ? .....): The following properties hold in $\mathcal{I}_{\sqsubset}(S_{\tilde{u}})$, if they hold in $\tilde{u}$-almost all of the S_i:

(i) $\mathcal{I}_{\sqsubset}(S_{\tilde{u}})$ satisfies the lattice equation $p=q$

(ii) " is totally ordered

(iii) " has exactly width n

(iv) " " " length n

(v) " " " breadth n

Proof. The only thing left to prove is, that $\mathcal{J}_{\sqsubset}(S_{\tilde{u}})$ has at least width (length, breadth) n if $\tilde{u}$-almost all of the S_i have. But having at least width (length, breadth) n may be expressed by an existential sentence. Hence the proof follows from 6.2.

BAKER and HALES ([1]) gave an example that the positive universal lattice sentences are not the only sentences preserved in passing from L to $\mathcal{J}(L)$.
In the case of sentences in the language of continuous lattices the situation is even less clear.

6.5) Question: What are the sentences of $\mathcal{L}_{CL}$ preserved in passing from the S_i to $\mathcal{J}_{\sqsubset}(S_{\tilde{u}})$?

We conclude with an example of such a sentence:
Let $\phi := \forall x \forall y \forall z \, ((x \sqsubset z \text{ AND } x \sqsubset y) \Rightarrow x \sqsubset y \wedge z)$.

6.6) Proposition: If $\tilde{u}$-almost all of the S_i satisfy ϕ, then so does $\mathcal{J}_{\sqsubset}(S_{\tilde{u}})$.
Proof. Clearly, if $\{i \in I;\ S_i \models \phi\}$ is contained in $\tilde{u}$, then $S_{\tilde{u}} \models \phi$. Now let $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \in \mathcal{J}_{\sqsubset}(S_{\tilde{u}})$ satisfy $\mathcal{J}_3 \ll \mathcal{J}_1$ and $\mathcal{J}_3 \ll \mathcal{J}_1$. Choose elements $a_1, b_1 \in \mathcal{J}_1$ and $a_2, b_2 \in \mathcal{J}_2$ such that $a_1 \sqsubset_{\tilde{u}} b_1$, $a_2 \sqsubset_{\tilde{u}} b_2$ and $\mathcal{J}_3 \subseteq \downarrow a_1$, $\mathcal{J}_3 \subseteq \downarrow a_2$. Then we have $\mathcal{J}_3 \subseteq \downarrow a_1 \wedge a_2$ and $a_1 \wedge a_2 \sqsubset_{\tilde{u}} b_1$, $a_1 \wedge a_2 \sqsubset_{\tilde{u}} b_2$. This imply $a_1 \wedge a_2 \sqsubset_{\tilde{u}} b_1 \wedge b_2$ as $S_{\tilde{u}}$ satisfies ϕ.
But this meas exactly $a_1 \wedge a_2 \in c(\mathcal{J}_1 \cap \mathcal{J}_2) = \mathcal{J}_1 \wedge \mathcal{J}_2$. Hence $\mathcal{J}_3 \subseteq \downarrow a_1 \wedge a_2 \subseteq \mathcal{J}_1 \wedge \mathcal{J}_2$, i.e. $\mathcal{J}_3 \ll \mathcal{J}_1 \wedge \mathcal{J}_2$.

6.7) Corollary: If every S_i is a distributive lattice such that the set $P(S_i)$ of prime elements of S_i is closed in the Lawson topology of S_i, then the same holds in $\mathcal{J}_{\sqsubset}(S_{\tilde{u}})$ for every $\tilde{u} \in \beta I$.
Proof. Use Corollary 6.4 and the fact that in a distributive continuous lattice L the set $P(L)$ is closed iff $L \models \phi$ (see [6]).

6.8) <u>Corollary</u> (i) If all the S_i satisfy ϕ, then so does $\amalg S_i$

(ii) If all the S_i satisfy a given lattice equation $p=q$, then so does $\amalg S_i$

(iii) If all the S_i are distributive and if $P(S_i)$ is always closed, then the same is true for $\amalg S_i$

Proof. (i) Let $p: \mathcal{E} \to \beta I$ the bundle constructed in section 5. Then $\amalg S_i \cong LC(p)$.

Let $\sigma \ll \tau_1, \tau_2$. By theorem 2.9 we may find continuous sections such that $\sigma \leq \gamma_1 \ll \tau_1$ and $\sigma \leq \gamma_2 \ll \tau_2$. Then $\gamma := \gamma_1 \wedge \gamma_2$ is continuous and satisfies $\sigma \leq \gamma \ll \tau_1, \tau_2$. Hence, we have $\gamma(\tilde{u}) \ll \tau_1(\tilde{u}), \tau_2(\tilde{u})$ for all $\tilde{u} \in \beta I$ and therefore $\gamma(\tilde{u}) \ll \tau_1(\tilde{u}) \wedge \tau_2(\tilde{u})$ by 6.6.

∴ by theorem 5.1 the set $\{u \in \mathcal{E};\ u \gg \gamma p(u)\}$ is open we get $G(\tau_1 \wedge \tau_2) \subseteq G(\gamma)^\circ$.

But this implies $\sigma \leq \gamma \ll \tau_1 \wedge \tau_2$.

(ii) Clear, because finite inf's and sup's are calculated pointwise

(iii) Clear by (i), (ii) and [6]