SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME(S) Jimmie Lawson	DATE	M 11	D 19	Y 76

TOPIC Non-continuous lattices

REFERENCE

I. Compact Convex Sets

Let V be a Hausdorff topological vector space and let K be a compact convex subset of K. As is well-known the space of all compact subsets of K, $\Gamma(K)$, is a compact space with respect to the Vietoris topology.

Let $c(K)$ denote the subspace of $\Gamma(K)$ of all compact convex subsets. It is not difficult to verify that $c(K)$ is closed and hence compact. $c(K)$ is a semilattice with respect to the operation $A \wedge B$ = closed convex hull of $A \cup B$. With respect to this operation $c(K)$ is a topological semilattice. (To see this, note that the function $F : \Gamma(K) \times \Gamma(K) \times \Gamma(I) \to \Gamma(K)$ defined by $F(A,B,T) = \{ta + (1-t)b : a \in A, b \in B, t \in T\}$ is continuous, and that for $A,B \in c(K)$, then $A \wedge B = F(A,B,I)$. Now if the empty set is joined discretely then $S = c(K) \cup \{\emptyset\}$ is a compact topological semilattice with identity.

Theorem 1. TAE

(1) S is a continuous lattice.

(2) K can be affinely and homeomorphically embedded in a locally convex topological vector space.

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

(3) Each point in K has a basis of (not necessarily open) neighborhoods in K which are convex.

<u>Proof</u>. The equivalence of (2) and (3) is the principal result of my "Embeddings of compact convex sets and locally compact cones", to appear Pac. J. Math.

Not (3) $\Rightarrow$ Not (1) is straight-forward since S fails to have small semilattices at any point $\{p\}$ for which p fails to have a basis of convex neighborhoods in K.

(2) $\Rightarrow$ (1) Let $A \in c(K)$, $A \neq \emptyset$, where K sits in a locally convex space V. Then $A = \bigcap \{ \overline{(A+U) \cap K} :$ U a convex neighborhood of $0\}$ and for each such U, $\overline{(A+U) \cap K} \ll A$. Hence S is a continuous lattice.

Until recently it was an open question whether there existed compact convex sets which failed to satisfy the conditions of Theorem 1. I have been interested in the problem for several years for two reasons:

(1) Such an example would give rise to a non-continuous compact topological semilattice.

(2) Such an example would form the base for a locally compact cone that could not be embedded in a locally convex topological vector space.

Last year J. W. Roberts, University of South Carolina, constructed an example of a compact convex set with no extreme points - in particular not embeddable in a locally convex space. ("A compact convex set with no extreme points"). I don't believe it has appeared yet, but I have a preprint of the paper. More recently he has shown something to the effect that for L_p, $0 < p < 1$, most compact convex sets are not embeddable (most meaning with respect to Baire category I think). I don't have this paper and am not sure of the exact formulation.

As a historical note, R. E. Jamison also discovered the equivalence of (1) and (2) in work on his dissertation at the University of Washington. We discussed the problem at the San Francisco AMS meeting some three years ago.

II. Schreier - Redei Extensions

Let S and T be commutative semigroups with identity. Let $f : T \times T \to S$ satisfy

A. The identity condition

$$f(x,1) = 1 = f(1,x)$$

B. The associativity condition

$$f(y,z)\cdot f(y,yz) = f(x,y)\cdot f(xy,z)$$

C. The commutativity condition

$$f(x,y) = f(y,x) .$$

Then we may form $S \circ T = S \times_f T$ with multiplication

$$(s_1,t_1)(s_2,t_2) = (s_1 s_2 f(t_1,t_2), t_1 t_2) .$$

The groupoid $S \circ T$ is actually a commutative semigroup with identity $(1,1)$, and we have an exact sequence of semigroups

$$S \xrightarrow{i} S \circ T \xrightarrow{\pi} T$$

where $i(s) = (s,1)$ and $\pi(s,t) = t$. A much more general treatment of these matters appears in L . Redei's "Die Verallgemeinerung der Schreierscher Erweiterungstheorie", Acta. Sci. Math. Szeged., 1952, pp. 252-273. John Ganci wrote his dissertation under my direction on the topological version of these results.

For semilattices S and T we wish to modify the construction slightly by no longer requiring that $f : T \times T \to S$ satisfy the identity condition. In this case $S \circ T$ is still a commutative semigroup, but $(1,1)$ need not be an identity. Also in this case $(s,t) \in S \circ T$ is idempotent $\Leftrightarrow s \leq f(t,t)$. Forming $S \circ T$ and taking $E(S \circ T)$, the idempotents, provides a useful construction for building examples of semilattices.

Suppose $g : T \to S$ is order preserving. Then $f : T \times T \to S$ defined by $f(x,y) = g(x \wedge y)$ satisfies the associativity and commutativity condition. In this case $E(S \circ T) = \{(s,t) : s \leq g(t)\}$. If S and T are continuous lattices and g is continuous, then it can be shown that $E(S \circ T)$ is a continuous lattice (with the subspace topology of $S \times T$) with identity $(g(1), 1)$.

The amazing thing is that $E(S \circ T)$ may be a compact topological semilattice with the subspace topology even though f is not continuous. My example in "Lattices with no interval homomorphisms" can be viewed in this light. Let S denote $[0,\infty]$ and $T = 2^{\omega}$. Then a function

$\sigma : 2^{\omega} \to [0,\infty]$ is constructed where $\sigma = \wedge \sigma_i$ such that for $f(x,y) = \sigma(x \wedge y)$, $E(S \times_f T)$ is a compact topological semilattice with identity $(\infty,(1))$, but such that $E(S \circ T)$ is not continuous. For the construction to work it appears that g must be "sufficiently" but not "excessively" discontinuous. Perhaps viewing the construction from this vantage point of Schreier extensions might be helpful for the construction of other examples; I'm not sure.

III. Conditions which imply a lattice is continuous.

We have four classes of lattices CL $\subset$ compact semilattices w . 1 $\subset$ meet-continuous complete lattices $\subset$ complete lattices, where a lattice is meet-continuous if $x_\alpha \nearrow x$ implies $x_\alpha \wedge y \nearrow x \wedge y$ ($x_\alpha \nearrow x$ if x_α is an increasing set with $\sup x_\alpha = x$).

Interesting problems arise by studying what sufficient conditions on a lattice in one class puts it in a smaller class. (E.g., if we call the classes A,B,C and D, B+ finite-dimensional Peano continuum $\Rightarrow$ A, D + $\forall x$, $x = \sup\{y : y \ll x\} \Rightarrow$ A .)

Let $^N 2$ denote the semilattice of all finite subsets of N under union. A set P in a semilattice is an <u>irredundant</u> <u>set</u> if for any two finite subsets $F_1, F_2 \subset P$, $\inf F_1 = \inf F_2$ implies $F_1 = F_2$. The singletons in $^N 2$ form an irredundant set, and it is easily seen that a semilattice has a countable irredundant set if and only if it has a semilattice isomorphic to N_2 . If neither of these conditions exist, we say S has <u>weak finite breadth</u>. Finite breadth implies weak finite breadth, but not conversely.

<u>Theorem</u> 2. If S is a complete meet-continuous lattice, $x \nleq y$, and $z \ll x$ implies $z \leq y$, then $\downarrow x \setminus \downarrow y$ contains a countable irredundant set.

<u>Proof</u>. First of all note that as a result of meet continuity $a \ll b \Leftrightarrow \sup D = b$ for some up-directed set implies $a \leq d$ for some $d \in D$.

Since not $x \ll x$, there exists a directed set D with $x = \sup D$, but $x \neq d \ \forall d \in D$. Pick $x_1 \in D$ such that $x_1 \nleq y$.

Suppose $A_K = \{x_1, \ldots, x_K\}$ has been chosen satisfying (i) A_K is irredundant, an (ii) the subsemilattice S_K generated by A_K is a subset of $\downarrow x \setminus \downarrow y$. Let $z = x_1 x_2 \ldots x_K$. Since not $z \ll x$, there exists a directed set D with $x = \sup D$, but $z \nleq d \ \forall d \in D$. For each $s \in S_K$, sD is a directed set converging up to s . Since S_K is finite, $\exists\, b \in D$ $sd \neq td$ if $s \neq t$ for all $s,t \in S_K$, for all $d \geq b$. Since $\sup zD = z$ and $z \nleq y$, there exists $p \geq b$, $p \in D$, such that $pz \nleq y$. Let $x_{K+1} = p$. Then it is easily verified that $\{x_1, \ldots, x_{K+1}\}$ is irredundant and the subsemilattice this set generates is a subset of $\downarrow x \setminus \downarrow y$. Hence by recursion there exists a set with the desired properties. □

Corollary 3. If S is a complete meet-continuous lattice of weak finite breadth, then S is a continuous lattice.

Proof. Let $x \in S$. Let $y = \sup\{z : z \ll x\}$. If $y < x$, then by Theorem 2, there exists a countable irredundant set in $\downarrow x \setminus \Downarrow y$. This is impossible since S has weak finite breadth. Hence $x = y$, and thus S is a continuous lattice. □