SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME(S)	HOFMANN	DATE M	D	Y
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TOPIC OBSERVATIONS

REFERENCE KEIMEL - MISLOVE SCS 9 - 30-76 (rec'd Tulane 11-12-76)

1) Keimel and Mislove establish in SCS 9-30 the following

LEMMA. Let $L \in \underline{CL}$ and S a closed subsemilattice ($1 \in S$!), $s \in L \setminus S$. Then there is a closed subsemilattice T of L such that

(i) $s \notin T$. (ii) $S \subseteq \operatorname{int} T$.

I feel the necessity to prove this again.

Proof. Since L has small subsemilattices and by Nachbin's theorem there is a closed convex subsemilattice neighborhood U of s with $U \cap S = \emptyset$. (Nachbin's theorem says in our context that the convex closed neighborhoods of s form a basis; we observe that the closed convex hull of a semilattice neighborhood is a semilattice neighborhood.)

Let $u = \min U$, and set $m = \min \uparrow u \cap S$. I claim there is an $a \ll m$ such that $\uparrow a \cap U = \emptyset$ (for if not, then for every $a \ll m$ we would have an $u_a \in \uparrow a \cap U$; if v is a cluster point of the net $(u_a)_{a \ll m}$ then $u \leq m \leq v$ with $u, v \in U$; by the convexity of U we would have $m \in U$, which would contradict $m \in S$ because of $U \cap S = \emptyset$). We observe

(1) If $t \in \uparrow u \cap S$; then $t \in \operatorname{int} \uparrow a$; also $s \notin \uparrow a$; indeed $\uparrow u \cap S \subseteq \uparrow m \subseteq \operatorname{int} \uparrow a$. The ~~last~~ second assertion is due to $\uparrow a \cap U = \emptyset$.

Let $I(u) = S \cap \uparrow u$. Then $s \notin S \setminus \operatorname{int} \uparrow u$,

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

since $s \in \text{int } U \subseteq \text{int } \uparrow u$. We record (in view of $I(u) \subseteq \text{int } I(u)^-$):

(2) If $t \in S \setminus \uparrow u$, then $t \in \text{int } I(u)^-$; also $s \notin I(u)^-$.

We now set $T = I(u)^- \cup \uparrow a$; since $I(u)^-$ is an ideal, this is a close subsemigroup. Because of (1) and (2) ~~xx~~ it satisfies the requirements. □

However, I wonder, whether this is the most economic approach to the

PROPOSITION (Keimel,Mislove). If $L \in \underline{CL}$, then $\dot{\Gamma}(L) = \{S \in \Gamma(L): 1 \in S = SS\}$ is a continuous lattice.

Proof. Let $A,B \in \dot{\Gamma}(L)$. Then AB is the smallest closed subsemilattice containing $A \cup B$: Indeed,if S is this subsemilattice, then $(A\cup B)^2 \subseteq S$, but $(A \cup B)^2 = A \cup B \cup AB = AB$ (since $A = A1 \subseteq AB$ and $B = 1B \subseteq AB$). On the other hand, AB is a subsemigroup containing A,B henc $A \cup B$, whence $S \subseteq AB$. Thus, ~~thx~~ while the (inf) operation on $\Gamma(L)$ is $(A,B) \longmapsto A \cup B$, the (inf) operation on $\dot{\Gamma}(L)$ is $(A,B) \longmapsto AB$. Now, when Schliemann excavated Troya he found a cerami tile in Troya 4 (fourth from top) on~~x~~ which there was engraved:"If thou takest a compact abelian monoid,then thou findst that the space of closed submonoids is compact and a topological monoid under the multiplication of sets." There is also a reference of that in Hofmos "Elements of Compact Semigroups" A.7.1 and B.4.2 .□

Let me notice that Keimel and Gierz utilized $\dot{\Gamma}(L)$ in "Topologische Darstellung von Verbänden " M.Z.150 (1976),p.90.

The Prpposition incidentally illustrates a situation in which for Lemma 1.13 in SCS -Darmstadt 8-1-76 we have a kernel operator $k: S \longrightarrow S$ (Scott continuous) such that the $\underline{CL}$ - topology of $k(S)$ is the one induced from that of S (which is not normally the case), while the inf operations of S and $k(S)$ still differ. ~~Hxr~~

Furthermore notice that L is a CL -retract of $\dot{\Gamma}(L)$, the retractio[n] being inf , the coretraction $s \mapsto \uparrow s$.

2) Proposition 2.1 in Keimel-Mislove is very nice and I wish Jimmie and I could have had that in the long ramblings of "irreducibility and generation" .

Condition (C) is something that is not entirely unfamiliar since ATLAS. Notice that (C) is equivalent to

(C') For each $x \in L$, int $\uparrow x$ is a filter.

Even in $\underline{Z}$, int $\uparrow x$ is often badly behaved (ATLAS ,Example 4.2) Notice that in a CL -object L with (C') for each $x \in L$ the element $\bar{x} = \inf \operatorname{int} \uparrow x$ is in A(L) (ATLAS 4.1 and Hofmann-Lawson, Irred 2.13). Clearly $x \mapsto \bar{x} : L \to L$ is monotone and idempotent and satisfies $x \leq \bar{x}$, hence is a closure operator, whose image is precisely A(L). Thus condition (C') is sufficient that A(L) is a complete lattice and indeed an inf subsemilattice. In the language of ATLAS Section 4 ~~[illegible]~~ ,this does not secure A(L)=B(L[)] but I wonder whether much of that Section does not apply here with A(L) in place of B(L). Does anyone have an example of an $L \in$ CL satisfying (C) [<=> (C')] with $L \neq A(L)$? Well,cancel the question and look at the standard Cantor semilattice. In that case A(L) is obtained by deleting all cofinite elements $\neq 1$ (formerly we said cocompact and therefore $A(L) \cong I$. Is it allright if a formulate a few ~~qu~~ questions at this point?

Q1. Let $L \in$ CL , $c: L \longrightarrow L$ a closure operator. When is $c(L) \in$ CL ? When is the corestriction $c': L \longrightarrow c(L)$ in CL ?

Note c'is right adjoint to the inclusion. It happens sometimes,but not too often, that a right adjoint is in CL. Q1 cannot be answered by just dualizing what we know about kernel operators, since the CL-theory is not inf-sup symmetric.

Q2. In the light of Q1 , when ~~xxxx~~ is ~~xxx~~ $A(L) \in \underline{CL}$ under the hypothesis (C)?

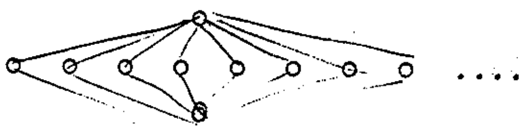
3) Re congruences on a CL-object

Perhaps there is a one page proof in pp.6-7 for a 20 page paper.

In another vein:

Let $L \in \underline{Z}$. Then (Cong* L , $\cap$) by HMS -Duality is isomorphic to $\dot{\sum}(K(L))$, the semilattice of sup- subsemilattices $A \subseteq K(L)$ containing 0 relative to the operation $(A,B) \longmapsto A \vee B$. Since Keimel and Mislove show that L has to be stably zero dimensional, if (Cong* L, $\cap$) $\in \underline{CL}$ we might just as well ask for a (discrete) semilattice S (with identity) such that the complete semilattice $\dot{\sum}S$ of all subsemilattices under the set product is a continuous lattice, and we may assume that S is without order dense chains.

I do not fully understand which $S \in \underline{S}$ qualify. I notice that $T \in K(\dot{\sum} S)$ if $S \setminus T$ is finite. Thus $\sum S$ is algebraic if every subsemilattice is the intersection of cofinite ones. Here is one that comes close but does not quite make it:



Here is one that does: S an arbitrary chain with maximum (and no order dense subchain if $\hat{S}$ is to be dimensionally stable)

4) Literature: Oswald Wyler, Compact complete lattices, Preprint 48 pp. (address Carnegie Mellon, Pittsburgh, Pa. 15213).

A categorist on the loose independently and without any reference known to any of you rediscovers CL à la Alan Day.