SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME(S)	DATE M	D	Y
Mislove	December	8	1976

TOPIC Closure operators and kernel operators in CL

REFERENCE
0. Observations by Hofmann on Keimis, Nov. 23,'76 (rec'd Dec. 1,'76)
1. Gierz, Hofmann, Keimel, Mislove SCS memo 8-1-1976

> "Silly ol' bear!" said Christopher Robin,
> "What are you doing? First you went around the tree twice by yourself, and then piglet ran after you, and you both went around together. And now, you were just about to go 'round a fourth time...."
> "Wait a minute!" said Pooh bear, ...
> "Ahh, I see. Yes. I've been foolish and deluded, and I'm a bear of no brain at all!"
>
> Winnie the Pooh

The reason for the quote is that it seemed particularly appropriate, since I'm now about to discuss closure operators and kernel operators once again. As of now, there have been no less than 3 memos on this subject; that's a fourth of the total number according to Hofmann's counting. Well, I guess there's no help for it. Here we go again!

I should perhaps comment that I have been giving a series of lectures in Salzmann's seminar here at Tübingen, which are concerned, naturally enough, with continuous lattices. The process of writing up what I am to talk about in each session has lead me to discover what I think is a unified way to approach these operators, and while the results about kernel operators are not new, the ones about closure operators are (at least to my knowledge). I was prompted to write this memo after reading the questions Q1 and Q2 of the reference, and I trust this will shed some light on that subject.

Having just received Scott's response to Hofmann's editorial, and not wishing to offend anyone, I guess I should comment that I view this memo as a prepreprint. This does bring up the fact that I would welcome comments about this with an eye toward improving the content and/or presentation. I will begin by quoting some salient facts from ATLAS.

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USA: U. California, Riverside (Stralka)
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Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

Note: After December 30, I will once again be back at Tulane. Thus all mail to me after this is received should be sent there. Thank you!

Definition 1. Let S and T be semilattices. A pair of order-preserving functions $g : S \to T$ and $d : T \to S$ is called a Galois connection between S and T if $d(g(s)) \leq s$ and $g(d(t)) \geq t$ for each $s \in S$ and each $t \in T$. Given such a pair, g is called the left adjoint of d, and d the right adjoint of g.

Proposition 2. Let S and T be semilattices, and let (g,d) be a Galois connection between S and T. Then $g(s) = \sup d^{-1}(\downarrow s)$ for each $s \in S$, and $d(t) = \inf g^{-1}(\uparrow t)$ for each $t \in T$. Thus g and d determine each other uniquely. Moreover, g preserves all existinf infima, and d preserves all existing suprema.

If S and T are complete lattices, and if $g : S \to T$ is an inf-preserving function, then the function $d : T \to S$ defined as above preserves all sups, and the pair (g,d) is a Galois connection between S and T. Dually, if $d : T \to S$ preserves all sups, then the function $g : S \to T$ defined as above preserves all infs, and (g,d) is a Galois connection between S and T.

Finally, if S and T are semilattices and (g,d) is a Galois connection between S and T, then the following are equivalent:

1. g is onto (one-to-one).
2. d is one-to-one (onto).

This, as indicated above, can be found in ATLAS; in any event, it's all quite straightforward to prove.

Proposition 3. Let S and T be compact Lawson semilattices, and (g,d) a Galois connection between S and T. The following are equivalent:

1. g is continuous.
2. For $t \ll t'$ in T, we have $d(t) \ll d(t')$ in S.

This is also from ATLAS; it's basically a corollary to Proposition 1.19, when one realizes that $x \in \operatorname{int} \uparrow y$ iff $y \ll x$ in a Lawson semilattice. Finally, we quote Proposition 1.26 (or a portion thereof).

Proposition 4. Let S and T be continuous lattices, and let $g : S \to T$ be a Scott continuous function. The following are then equivalent:

1. g preserves arbitrary infima.
2. $g : S \to T$ is a homomorphism of compact Lawson semilattices, if S and T are endowed with the Lawson topology.

Definition 5. Let CL denote the category whose objects are continuous lattices, and whose morphisms are Scott continuous functions which preserve arbitrary infima. Let CS denote the category of compact Lawson semilattices and continuous semilattices homomorphisms.

Theorem 6. Let $F : \underline{CS} \to \underline{CL}$ be the functor which assigns to each compact Lawson semilattice S the underlying continuous lattice, and to each CS-morphism f the CL-morphism f, and let $G : \underline{CL} \to \underline{CS}$ be the functor which assigns to each continuous lattice L the compact Lawson semilattice gotten by equipping L with the Lawson topology, and to each CS-morphism f, the CL-morphism f. Then CS and CL are isomorphic under the functors F and G.

Finally, we come to something which is not in ATLAS (at least I don't think it is).

Proposition 7. Let L be a continuous lattice, and $L' \subseteq L$ an inf-subsemilattice of L which contains the identity. The following are equivalent:

1. L' is a continuous lattice, and the natural injection $i : L' \to L$ is a CL-morphism.
2. L' is closed in L under arbitrary infs and up-directed sups.

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Proof. Clearly 1 implies 2 by the definition of a CL-morphism. Conversely, suppose that L' is closed in L under all infs and up-directed sups. We first show that L' is a continuous lattice. Indeed, since L' has arbitrary infs and contains the identity, for each $X \subseteq L'$, we define $\sup X = \inf \{y \in L' : x \leq y \text{ for each } x \in X\}$, and note that this is well-defined, and so L' is a complete lattice. Let $x \in L'$, and let $y \in L$ with $y \ll x$ in L. Let $y' = \inf (\uparrow y \cap L')$. We claim $y' \ll x$ in L': Let $D \subseteq L'$ be up-directed with $x \leq \sup D$. Then since L' is closed under up-directed sups, sup D is the same in L and L'. Hence, $y \ll x$ implies that there is some $d \in D$ with $y \leq d$. Since $d \in D \subseteq L'$, it then follows that $y' \leq d$. This establishes the claim. Now, each $x \in L'$ is the sup of those $y \in L$ with $y \ll x$ in L, and we have just shown that for each $x \in L'$ and each $y \in L$ with $y \ll x$ in L, there is some $y' \in L'$ with $y \leq y'$ and $y' \ll x$ in L'. It then follows that $x = \sup \{y' \in L' : y' \ll x \text{ in } L'\}$. Thus L' is a continuous lattice. The fact that L' is closed in L under arbitrary infs and up-directed sups then implies that the natural map $i : L' \to L$ is a CL-morphism.

Definition 8. Let L be a complete lattice. An order-preserving function $c : L \to L$ is a closure operator on L if $x \leq c(x) = c^2(x)$ for each $x \in L$. Dually, an order-preserving function $k : L \to L$ is a kernel operator on L if $k^2(x) = k(x) \leq x$ for each $x \in L$. If L is a continuous lattice, then by a continuous closure (kernel) operator on L we shall mean one that is Scott continuous.

Proposition 9. Let L be a complete lattice, c a closure operator on L, and k a kernel operator on L. Then c(L) and k(L) are complete lattices, c preserves all sups, and k preserves all infs. Moreover, $c : L \to c(L)$ is right adjoint to the inclusion map $i : c(L) \to L$, and $k : L \to k(L)$ is left adjoint to the inclusion map $i : k(L) \to L$. Hence c(L) is closed in L under arbitrary infs, and k(L) is closed in L under arbitrary sups.

Conversely, suppose that L is a complete lattice, and $L' \subseteq L$ is a subset containing the identity which is also a complete lattice, and such that $\inf_{L'} X = \inf_L X$ for each subset $X \subseteq L'$. Then the function $c : L \to L$ given by $c(x) = \inf (\uparrow x \cap L')$ is a closure operator on L with image L', and is therefore right adjoint to the inclusion map $i : L' \to L$. Dually, if $L'' \subseteq L$ is a subset containing the zero which is also a complete lattice, and if $\sup_{L''} X = \sup_L X$ for each subset $X \subseteq L''$, then the map $k : L \to L$ defined by $k(x) = \sup (\downarrow x \cap L'')$ is a kernel operator on L with image L", and so k is left adjoint to the inclusion $i : L'' \to L$.

Proof. Suppose that $c : L \to L$ is a closure operator on the complete lattice L, and let $X \subseteq L$ with $s = \sup X$. Then, since c preserves the order, $\sup c(X) \leq c(s)$. On the other hand, suppose that $y \in c(L)$ is an upper bound in c(L) for c(X). Then, for each $x \in X$, we have $x \leq c(x) \leq y$, and so y is an upper bound for X in L. Hence $s \leq y$, and so $c(s) \leq c(y) = y$. Thus $c(\sup X) = \sup c(X)$, so c preserves all sups. Since c(L) is then a complete sup-semilattice and c(0) is a zero for c(L), it follows that c(L) is a complete lattice. Since $c : L \to c(L)$ preserves arbitrary sups, its right adjoint $j : c(L) \to L$ preserves all infs, by Proposition 2. We show that j is the inclusion map: Indeed, the definition of j (Proposition 2) yields that $j(y) = \sup c^{-1}(y)$ for each $y \in c(L)$, since c is onto. Since $c(y) = y$, it follows that $y \leq j(y)$. On the other hand, $c(j(y)) = y$ as c preserves arbitrary sups, and so $j(y) \leq c(j(y)) = y$, the inequality following from the fact that c is a closure operator. Thus $y = j(y)$, and we have our claim. It then follows that c(L) is closed in L under arbitrary infs since j preserves all infs.

A dual argument shows that k preserves all infs, and that k is left adjoint to the inclusion map $i : k(L) \to L$, so that k(L) is closed in L under arbitrary sups, for each kernel operator k on L.

Now, suppose that $L' \subseteq L$ contains the identity, is a complete lattice whose infs agree with those in L. It is then clear that the map $c : L \to L$ given by $c(x) = \inf (\uparrow x \cap L')$ is a well-defined order-preserving function which also satisfies $x \leq c(x) = c^2(x)$ for each $x \in L$, whence c is a closure operator. It is also clear that $c(L) = L'$, and that $c(x) = x$ for each $x \in L'$. Hence c is right adjoint to the inclusion by the first part of the Proposition. The argument for the sup-semilattice L" follows dually.

as maps onto the image!

We should note that, since closure operators preserve all sups, they are always Scott continuous if they are defined on a continuous lattice. This Proposition then shows that, in order to consider a closure operator, it is equivalent to consider a subset L' of the lattice which contains the identity and is a complete lattice whose infs agree with those in the containing lattice. The following then completes the picture (to some extent) for continuous lattices.

<u>Proposition 10</u>. Let L be a continuous lattice, and $c : L \to L$ a closure operator on L. The following are equivalent:

1. $c(L)$ is a continuous lattice and the natural injection $i : c(L) \to L$ is a <u>CL</u>-morphism.
2. For $x,y \in L$, if $x \ll y$, then $c(x) \ll c(y)$.

Proof. First note that, in any case, c is the right adjoint of i by Proposition 9. Now, if 1 holds, then i is a homomorphism of compact semilattice by Proposition 4, and so c satisfies 2 by Proposition 3.

Conversely, suppose that 2 is satisfied. Then, since c preserves arbitrary sups, it follows that $c(L)$ is a continuous lattice. Moreover, Proposition 3 shows that i is a homomorphism of compact semilattices, and so i is a <u>CL</u>-morphism by Theorem 6.

<u>Example 11</u>. We now give some examples that seem to have a bearing on this situation.

a. Consider the closure operator c $I^2 \to I^2$ (I the unit interval) given by $c(x,y) = (x,y)$ if $x = 0$ and $y < 1/2$, and $c(x,y) = (1, \max(y,1/2))$ otherwise. It is readily verified that c is a closure operator, and that $c(I^2)$ is isomorphic to the unit interval. However, since c doesn't satisfy 2 above, $c(I^2)$ is not a <u>CL</u>-subobject of I^2.

b. Again we pick on I^2. This time we define $c : I^2 \to I^2$ by $c(x,y) = (x,y)$ if $x = 0$ and $y < 1/2$, and $c(x,y) = (1,y)$ otherwise. Again we leave it to whoever wants to to show that c is a closure operator. However, $c(I^2)$ is not a continuous lattice. Indeed, suppose that $y < 1/2$, and that $z \leq y$. We show that $(1,z) \not\ll (1,y)$: If $X = \{(0,w) : w < 1/2\}$, then X is an up-directed subset of $c(I^2)$ with $\sup X = (1,1/2)$ (this sup is in $c(I^2)$). Hence $(1,y) < \sup X$, but $(1,z) \not\leq (0,w)$ for any $w < 1/2$. It is then clear that each element $(1,y)$ in $c(I^2)$ with $y < 1/2$ is not the sup of the points of $c(I^2)$ which are way-below it in $c(I^2)$. Hence $c(I^2)$ is not a continuous lattice.

It should be noticed that $c(I^2)$ is not lower continuous in this second example, where, recall that a complete lattice is <u>lower continuous</u> if for each up-directed set X and each point x with $x \leq \sup X$, we have $x = \sup xX$. This motivates the following:

Conjecture: Let L be a continuous lattice and $c : L \to L$ a closure operator on L. The following are equivalent:

1. c(L) is a continuous lattice.
2. c(L) is a lower continuous complete lattice.

Of course the direction 1 implies 2 is true, since any continuous lattice is lower continuous, and so it is the converse that is unsettled.

For completeness sake, we record the following lemma; it was first formulated in "A lemma on primes appearing in algebra and analysis", Gierz and Keimel (to appear in Houston Journal), and later reformulated in reference 1.

Lemma 12. Let L be a continuous lattice, and $k : L \to L$ a kernel operator. The following are then equivalent:

1. k(L) is a continuous lattice and $k : L \to k(L)$ is a CL-morphism.
2. k preserves sups of up-directed sets.
3. k(L) is a continuous lattice and the natural injection $i : k(L) \to L$ preserves up-directed suprema.

Example 13. We here repeat two examples from Hofmis SCS memo 8-4-76 which are analogous to those presented above for closure operators.

a. Let $I = [0,1]$ be the unit interval under min multiplication, and define $k : I \to I$ by $k(0) = 0$, and $k(x) = 1$ for $x \neq 0$. Then k is a kernel operator on L, and k(L) is a continuous lattice. However, k is not a CL-morphism.

b. This time we let $k : I^2 \to I^2$ be defined by $k(x,y) = (x,y)$ if $\max\{x,y\} = 1$, and $k(x,y) = 0$ otherwise. Then k is a kernel operator on I^2. However, k does not preserve sups of up-directed sets, and $k(I^2)$ is not a continuous lattice.

Given our conjecture about when c(L) is a continuous lattice for a closure operator on the continuous lattice L, and in light of these two examples, one might be lead to conjecture that k(L) is a continuous lattice if and only if k(L) is lower continuous for a kernel operator on a continuous lattice L. The following shows that this is not the case:

Example 14. This is taken from a yet to be written SCS memo (dated 12-32-76 ?) by Klaus Keimel and myself about the complete lattice of open subsets of a topological space.

Let D be the open unit disk in the complex plane together with one point on the boundary, say (1,0). Then, as we all know, D is a Hausdorff space in the relative topology which is not locally compact. There is a Theorem due to Isbell that says that the open sets of a Hausdorff space form a continuous lattice if and only if the space is locally compact, and so we know that O(D) is not a continuous lattice. Now, the lattice 2^D of all subsets of D is an algebraic lattice, and the map $k : 2^D \to 2^D$ given by k(X) = int X, the interior of the set X, for each X   D, is a kernel operator on 2^D with image O(D). Moreover, since sups in O(D) are just unions it is clear that O(D) is lower continuous, being a sublattice of 2^D which is closed under arbitrary suprema. However, as we remarked above, O(D) is not a continuous lattice.

We can record the following partial result in this direction.

Lemma. Let L be a continuous lattice, and $k : L \to L$ a kernel operator on L. If k(L) is closed in L under arbitrary infs, then k(L) is a continuous lattice.

Proof. Since k is a kernel operator, Proposition 9 shows that k(L) is closed in L under arbitrary sups; if k(L) is also closed in L under arbitrary infs, then Proposition 7 shows that k(L) is a CL-subobject of L.

Note that the example O(D) shows that we cannot weaken the hypothesis of the lemma to k(L) being closed in L under finite infs.

The following corollary to Lemma 12 shows that CL is closed under quotients.

Proposition 15. Let L be a continuous lattice, L' a complete lattice, and $f : L \to L'$ a function which preserves all infs and all up-directed sups. Then f(L) is a continuous lattice, and $f : L \to f(L)$ is a CL-surmorphism.

Proof. Let $f : L \to L'$ be as hypothesised. Then f is an inf-preserving morphism between complete lattices, and so f has a sup-preserving right adjoint $j : L' \to L$. Moreover, Proposition 2 shows that $j(y) = \inf f^{-1}(\uparrow y)$, and so $j(y) = \inf f^{-1}(y)$ for each $y \in f(L)$. Let $k = j \cdot f : L \to L$. Then clearly $k(x) \leq x$ for each $x \in L$; moreover, $k^2(x) = j(f(j(f(x)))) = j(f(x)) = k(x)$, since $f(j(f(x))) = f(\inf f^{-1}(f(x))) = \inf f(f^{-1}(f(x))) = f(x)$, the second equality following from the inf-preservation of f. Thus k is a kernel operator on L. Moreover, f preserves up-directed sups by hypothesis, and j preserves all sups, being a right adjoint; it therefore follows that k preserves sups of up-directed sets, and so k(L) is a continuous lattice by Lemma 12. Now, we have already shown that $f(j(f(x))) = f(x)$ for each $x \in L$. It then follows that $f \cdot j$ is the identity on f(L) and $j \cdot f$ is the identity on k(L). Since f and j are both order-preserving, it follows that k(L) and f(L) are lattice isomorphic under the pair of isomorphisms f and j (or, more correctly, their restrictions and corestrictions to f(L) and k(L)). Finally, since f preserves all infs and up-directed sups, the map $f : L \to f(L)$ is a <u>CL</u>-surmorphism.

We note that a corollary of this is that the category <u>CS</u> is closed under quotients.

<u>Proposition 16</u>. Let L be a complete lattice, and $c : L \to L$ a closure operator on L. The following are equivalent:

1. c preserves all infs.

2. The complete lattice $c(L)$ is a complete inf-subsemilattice of L and the map $c : L \to c(L)$ is a quotient in the category of complete lattices and complete lattice homomorphisms (i.e., ones which preserve all infs and sups).

3. The map $k : L \to L$ given by $k(x) = \inf c^{-1}(c(x))$ is a kernel operator on L which preserves all sups, the maps $k \cdot c : k(L) \to k(L)$ and $c \cdot k : c(L) \to c(L)$ are both the identity map, whence $k(L)$ and $c(L)$ are lattice isomorphic, and $c(x) = \sup k^{-1}(k(x))$ for each $x \in L$.

Dually, if $k : L \to L$ is a kernel operator on L, then the following are equivalent:

a. k preserves all sups.

b. The complete lattice $k(L)$ is a complete sup-subsemilattice of L and the map $k : L \to k(L)$ is a quotient in the category of compete lattices and complete homomorphisms.

c. The map $c : L \to L$ given by $c(x) = \sup k^{-1}(k(x))$ is a closure operator on L which preserves all infs, the maps $c \cdot k : c(L) \to c(L)$ and $k \cdot c : k(L) \to k(L)$ are both the identity map, whence $k(L)$ and $c(L)$ are lattice isomorphic, and $k(x) = \inf c^{-1}(c(x))$ for each $x \in L$.

Finally, if L' is a complete lattice and $f : L \to L'$ a homomorphism preserving all infs and sups, then the map $c : L \to L$ given by $c(x) = \sup f^{-1}(f(x))$ is a closure operator on L which preserves all infs, the map $k : L \to L$ given by $k(x) = \inf f^{-1}(f(x))$ is a kernel operator on L which preserves all sups, the operators c and k are related as above, and the lattices $c(L)$, $k(L)$, and $f(L)$ are all isomorphic.

Proof. We shall first show that 1 thru 3 are equivalent; the equivalence of a thru c follows by dual arguments. Since closure operators preserve all sups, 1 and 2 are equivalent in light of Proposition 9. We show that 1 and 3 are also equivalent. Let c be a closure operator on the complete lattice L which preserves all infs, and define $k : L \to L$ by $k(x) = \inf c^{-1}(c(x))$. Then $k(x) \leq x$ is clear; since c preserves all infs, we have $c(k(x)) = c(x)$ for each $x \in L$, and so $k^2(x) = \inf c^{-1}(c(k(x)) = \inf c^{-1}(c(x)) = k(x)$, so that k is indeed a kernel operator on L. If we let $j : L \to L$ be the right adjoint of c (which exists as c preserves all infs), then Proposition 9 shows that $j(y) = \inf c^{-1}(\uparrow y)$ for each $y \in L$, and so $k = j \cdot c$. Now, j preserves all sups since it is a right adjoint, and c does also for the same reason. Hence k also preserves all sups. We have already seen that $c(k(x)) = c(x)$ for each $x \in L$; also, $k(c(x)) = \inf c^{-1}(c(c(x))) = \inf c^{-1}(c(x)) = k(x)$ for each $x \in L$, since c is idempotent. Thus $c(k(c(x))) = c^2(x) = c(x)$ and $k(c(k(x))) = k^2(x) = k(x)$ for each $x \in L$, and so we have that $k \cdot c : k(L) \to k(L)$ and $c \cdot k : c(L) \to c(L)$ are both the identity. Since c and k are both order-preserving, it follows that the restrictions and corestrictions of c and k to k(L) and c(L) give lattice isomorphisms between them. Finally, let $x \in L$. If $y \in k^{-1}(k(x))$, then $k(y) = k(x)$, and so $c(y) = c(k(y)) = c(k(x)) = c(x)$. Hence, $c(\sup k^{-1}(k(x))) = \sup c(k^{-1}(k(x))) = \sup c(x) = c(x)$. This shows that for $s = \sup k^{-1}(k(x))$, we have $c(s) = c(x)$. But, $s \leq c(s)$, and $k(c(s)) = k(s) = k(x)$, whence $s = c(s)$ by the definition of s. Therefore $c(x) = \sup k^{-1}(k(x))$ as claimed.

Now, we suppose that L' is a complete lattice, and $f : L \to L'$ a complete lattice homomorphism. Define $c : L \to L$ by $c(x) = \sup f^{-1}(f(x))$. Then $x \leq c(x)$ is clear, and since f preserves all sups, we have $f(c(x)) = f(x)$ for each x, whence it follows that $c^2(x) = c(x)$ for each $x \in L$. Thus c is a closure operator on L. If $j : L' \to L$ is the left adjoint of f (which exists as f preserves all sups), then we see that $c = j \cdot f$, and so c preserves all infs. A dual argument shows that $k : L \to L$ given by $k(x) = \inf f^{-1}(f(x))$ is an inf-preserving kernel operator on L. Let $x \in L$. then, $y \in c^{-1}(c(x))$ if and only if $c(y) = c(x)$ iff $\sup f^{-1}(f(y)) = \sup f^{-1}(f(x))$ iff $f(y) = f(x)$ (since f preserves all sups) iff $y \in f^{-1}(f(x))$. Thus, $c^{-1}(c(x)) = f^{-1}(f(x))$, and so $k(x) = \inf f^{-1}(f(x)) = \inf c^{-1}(c(x))$. A similar argument shows that $c(x) = \sup k^{-1}(k(x))$ for each $x \in L$. It then follows that c and k are related as indicated, and so the maps $k \cdot c : k(L) \to k(L)$ and $c \cdot k : c(L) \to c(L)$ are both the identity map, whence k(L) and c(L) are lattice isomorphic. Lastly, we have already noted that $c = j \cdot f$, and it then follows that $j : f(L) \to c(L)$ is one-tone and onto. Since j also preserves the order, j is a complete lattice isomorphism.

Remark: One of the things that emerges from the above Proposition is that kernel operators arise from inf-preserving homomorphisms on the complete lattice L, and closure operators arise from sup-preserving homomorphisms on the lattice L. This then makes it clear why kernel operators are of more interest for continuous lattices than closure operators, if it were not already clear.

We note additionally that the hypothesis that c preserve all infs in part one of the Proposition is necessary. Indeed, we let L = I be the unit interval, and define $c : L \to L$ by $c(0) = 0$, and $c(x) = 1$ if $x \neq 0$. Then c is a closure operator on L which preserves finite infs, but c does not preserve all infs. I guess an interesting question at this point would be to try and classify those closure operators which preserve finite infs; I have not looked at this.

Proposition 17. Let L be a continuous lattice, and $c : L \to L$ a closure operator on L. The following are equivalent:

1. $c(L)$ is a continuous lattice and $c : L \to c(L)$ is a CL-surmorphism which preserves all infs and all sups.

2. The map $k : L \to L$ given by $k(x) = \inf c^{-1}(c(x))$ is a continuous kernel operator on L which preserves all sups, $c(L)$ and $k(L)$ are lattice isomorphic under the restrictions and corestrictions of c and k to $k(L)$ and $c(L)$, and $c(x) = \sup k^{-1}(k(x))$ for each $x \in L$.

3. c preserves all infs.

Dually, if $k : L \to L$ is a kernel operator on L, then the following are equivalent:

a. $k(L)$ is a continuous lattice and $k : L \to k(L)$ is a CL-surmorphism which preserves all sups and infs.

b. The map $c : L \to L$ given by $c(x) = \sup k^{-1}(k(x))$ is a closure operator on L which preserves all infs, $c(L)$ and $k(L)$ are lattice isomorphic under the restrictions and corestrictions of c and k to $k(L)$ and $c(L)$, and $k(x) = \inf c^{-1}(c(x))$.

c. k preserves all sups.

Finally, if L' is any complete lattice and $f : L \to L'$ a complete lattice homomorphism, then the map $c : L \to L$ given by $c(x) = \sup f^{-1}(f(x))$ is a closure operator on L which preserves all infs, the map $k : L \to L$ given by $k(x) = \inf f^{-1}(f(x))$ is a continuous kernel operator on L which preserves all sups, the operators k and c are related as above, and the continuous lattices $c(L)$, $k(L)$, and $f(L)$ are all isomorphic.

Proof. After Proposition 16, we really don't have that much to prove. Indeed, the equivalence of 1 and 3 follows from Proposition 9 and the definition of a CL-morphism, and that of 1 and 2 follows from the equivalence of 1 and 2 in Proposition 16. The equivalence of a thru c can also be got at in this fashion. The final remark also follows from the final remark in Proposition 16. (We note that a kernel operator which preserves all sups is automatically continuous.)

Remark: The point of this proposition is that closure operators on continuous lattices which preserve all infs arise naturally from complete lattice surmorphisms in the category CL, and likewise for kernel operators which preserve all sups. Also note that the hypotheses on f and L' in the final remark are equivalent to assuming that $f(L)$ is a continuous lattice and $f : L \to f(L)$ is a complete lattice surmorphism in CL.

Finally, one wight wonder about the question of a kernel operator preserving the way-below relation, which would seem to be dual to a closure operator preserving

Proposition 17. Let L be a continuous lattice, and $c : L \to L$ a closure operator on L. The following are equivalent:

1. $c(L)$ is a continuous lattice and $c : L \to c(L)$ is a CL-surmorphism which preserves all infs and all sups.

2. The map $k : L \to L$ given by $k(x) = \inf c^{-1}(c(x))$ is a continuous kernel operator on L which preserves all sups, $c(L)$ and $k(L)$ are lattice isomorphic under the restrictions and corestrictions of c and k to $k(L)$ and $c(L)$, and $c(x) = \sup k^{-1}(k(x))$ for each $x \in L$.

3. c preserves all infs.

Dually, if $k : L \to L$ is a kernel operator on L, then the following are equivalent:

a. $k(L)$ is a continuous lattice and $k : L \to k(L)$ is a CL-surmorphism which preserves all sups and infs.

b. The map $c : L \to L$ given by $c(x) = \sup k^{-1}(k(x))$ is a closure operator on L which preserves all infs, $c(L)$ and $k(L)$ are lattice isomorphic under the restrictions and corestrictions of c and k to $k(L)$ and $c(L)$, and $k(x) = \inf c^{-1}(c(x))$.

c. k preserves all sups.

Finally, if L' is any complete lattice and $f : L \to L'$ a complete lattice homomorphism, then the map $c : L \to L$ given by $c(x) = \sup f^{-1}(f(x))$ is a closure operator on L which preserves all infs, the map $k : L \to L$ given by $k(x) = \inf f^{-1}(f(x))$ is a continuous kernel operator on L which preserves all sups, the operators k and c are related as above, and the continuous lattices $c(L)$, $k(L)$, and $f(L)$ are all isomorphic.

Proof. After Proposition 16, we really don't have that much to prove. Indeed, the equivalence of 1 and 3 follows from Proposition 9 and the definition of a CL-morphism, and that of 1 and 2 follows from the equivalence of 1 and 2 in Proposition 16. The equivalence of a thru c can also be got at in this fashion. The final remark also follows from the final remark in Proposition 16. (We note that a kernel operator which preserves all sups is automatically continuous.)

Remark: The point of this proposition is that closure operators on continuous lattices which preserve all infs arise naturally from complete lattice surmorphisms in the category CL, and likewise for kernel operators which preserve all sups. Also note that the hypotheses on f and L' in the final remark are equivalent to assuming that $f(L)$ is a continuous lattice and $f : L \to f(L)$ is a complete lattice surmorphism in CL.

Finally, one wight wonder about the question of a kernel operator preserving the way-below relation, which would seem to be dual to a closure operator preserving all infs. If one considers the kernel operator $k : I^2 \to I^2$ given by $k(x,y) = 0$ if $\max \{x,y\} < 1$, and $k(x,y) = (x,y)$ otherwise, then it is readily apparent that k preserves the way-below relation and also finite sups. However $k(I^2)$ is not a continuous lattice.

<u>Proposition 18</u>. Let L be a continuous lattice. Then the family cl(L) of all closure operators on L forms a continuous lattice in the pointwise operations.

Proof. We know that any product of continuous lattices is another such, and so L^L is a continuous lattice. We show that cl(L) is closed in L^L under all infs and up-directed sups; Proposition 7 then gives the desired result.

Let $C \subseteq cl(L)$ and let $s = \inf C$ in L^L. If $x \in L$, then $x \leq c(x)$ for each $c \in C$ implies that $x \leq s(x)$. Hence, $s(x) \leq s(s(x)) = s^2(x)$. Moreover, $s^2(x) = \inf_C \inf_C c(c'(x)) \leq \inf_C c^2(x) = s(x)$ s each $c \in C$ is idempotent. ~~On the other hand, if $c, c' \in C$, then $x \leq c'(x)$ implies $c(x) \leq c(c'(x))$, and so $c^2(x) = c(x) \leq c(c'(x))$. Thus $s(x) = \inf_C c(x) \leq \inf_C \inf_C c(c'(x))$~~ $= s^2(x)$, and so we conclude that $s^2(x) = s(x)$. Thus $s \in cl(L)$, and cl(L) is closed under all infs. Again let $C \subseteq cl(L)$, and this time assume that C is up-directed and that $s = \sup C$. Then $x \leq c(x)$ for all $c \in C$ implies that $x \leq s(x)$. Moreover, $s^2(x) = \sup_C \sup_C c(c'(x)) \geq \sup_C c^2(x) = s(x)$. Conversely, if $c, c' \in C$, then there is some $c'' \in C$ with $c, c' \leq c''$. Hence $c(c'(x)) \leq c''(c''(x)) \leq s(x)$, so $s^2(x) \leq s(x)$. Thus $s \in cl(L)$ once again, and so cl(L) is closed in L^L under up-directed sups.

<u>Remarks</u>: First, I should acnowledge that Dana Scott showed me a slick proof that cl(L) is a continuous lattice when we were in Oberwohlfach, and although I have forgotten the exact method, undoubtedly it is reflected here (this could even be the same proof).

Secondly, we could view the above proof as using closure operators to show that the family of closure operators on a continuous lattice forms a continuous lattice, for the operator $C : L^L \to L^L$ given by $C(f) = \inf \{ c \in cl(L) : f \leq c\}$ is a closure operator on L^L with image cl(L), and, according to Proposition 9, to show that cl(L) is a <u>CL</u>-subobject of L^L is equivalent to showing that the natural injection of cl(L) into L^L preserves all infs and all up-directed sups, which we have done. Note that, according to Proposition 10, this is also equivalent to showing that $f \ll g$ in L^L implies $C(f) \ll C(g)$ in cl(L), but this would mean calculating the way-below relation on cl(L), which I have studiously avoided (!).

Finally, we can give an explicit description of the operator C on L^L: Namely, for each $f \in L^L$, it is easily (?) verified that $C(f)(x) = \inf (\uparrow x \cap f(L))$.

Finally, it has just occured to me that, beginning with Proposition 16, I forgotten to show that various operators which I asserted were closure and/or kernel operators are actually monotone. This holds in each case; it was just an oversight that I'm too lazy to correct at this late moment. I should have pointed this out earlier for those of you who are reading the proofs, but if I had had the foresigt to do that, I wouldn't have overlooked the error in the first place, would I?