SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC The lattice of open subsets of a topological space+

REFERENCE
1. Gierz and Keimel, "A lemma on primes...", Houston Jour., to appear.
2. J. Lawson, "Intrinsic topologies..." Pac. Jour. 44(1973), 593-602.

If X is a topological space, then the space of open subsets of X, O(X), is a complete lattice. This memo is intended to give some results about when O(X) is a continuous lattice or a compact semilattice. These results are not all new, and they are not exhaustive; however, we hope they will shed some light on the problem, and eventually lead to a solution of it.

If we denote by 2^X the complete algebraic lattice of all subsets of X, then there is a natural kernel operator $k : 2^X \to 2^X$ with image O(X), namely, k(A) = int A, the interior of the set A. The following lemma shows that the now well-known lemma in reference 1 is of virtually no use in determining when O(X) is a continuous lattice.

Lemma 1. Let X be a T_1 space, and define $k : 2^X \to 2^X$ by k(A) = int A. If k preserves sups of up-directed sets, then X is discrete.

Proof. $X = \sup \{ F : F \subseteq X \text{ is finite}\}$, and this is an up-directed sup. Hence, if k preserves up-directed sups, we have $X = \sup \{ k(F) : F \subseteq X \text{ is finite}\}$. Thus, if $x \in X$, then there is some $F \subset X$ finite with $x \in k(F)$, and k(F) is a finite open set. Since X is T_1, points are closed, and so it follows that each point of k(F) is open. Therefore {x} is open, and so X is discrete.

As a result of this lemma, we see that whether or not O(X) is a continuous lattice must be determined independently of the lattice 2^X; thus the way-below relation on O(X) must be determined.

Definition 2. Let L be a complete lattice. For $x,y \in L$, we write $x \ll y$ if and only if for each up-directed set $A \subset L$ with $y \leq \sup A$, there is some $a \in A$ with $x \leq a$. We write $x \lll y$ if and only if, for each up-directed subset A of L with $y \leq \sup A$, there is some $a \in A$ with $x \ll a$.

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+: This memo stems from conversations held in Darmstadt in September; thanks to A v H

Definition 3. Let X be a topological space, $U \subset V$ open subsets of X. We say U is relatively compact in V if each open cover of V admits a finite subcover of U. Clearly U is relatively compact in V if and only if $U \ll V$ in $O(X)$.

Proposition 4. Let X be a Hausdorff space, and let A, B be open subsets of X. The following are equivalent:

1. $A \lll B$ in $O(X)$.
2. $\overline{A} \subset B$ and $\overline{A}$ is compact.

Proof. Suppose that $A \ll B$ in $O(X)$, and let $x \in X \setminus B$. Then X Hausdorff implies that the family of closed neighborhoods of x is downwards directed and has intersection $\{x\}$, and so the family $\{B \setminus N : N$ is a closed neighborhood of $x\}$ is an up-directed family in $O(X)$ whose sup is B. $A \ll B$ then implies that there is some closed neighborhood N with $A \subset B \setminus N$. It follows that $\overline{A} \subset B$. Since $A \lll B$ implies $A \ll B$, we have $A \lll B$ implies that $\overline{A} \subset B$. Second, assume that $A \lll B$, and let $\{O_i\}$ be an open cover of $\overline{A}$. Then, the family $\{O_i \cup B \setminus \overline{A}\}$ is an open cover of B, and since $A \lll B$, it follows that $A \ll O_i \cup B \setminus \overline{A}$ for some i, if we assume that the O_i are up-directed, which is possible by taking finite unions of the O_i's if necessary. Then, the first part of the proof shows that $\overline{A} \subset O_i$; this then demonstrates the compactness of $\overline{A}$. Hence we have shown that 1 implies 2.

Conversely, it is clear from the definitions that $\overline{A} \subset B$ and $\overline{A}$ compact imply A is relatively compact in B, and so $A \ll B$. Hence, if $\{O_i\}$ is any up-directed family of open subsets of X with $B \leq \sup O_i$, then $\overline{A} \subset \bigcup O_i$, and so there is some i with $\overline{A} \subset O_i$. But, the comment just made then implies that $A \ll O_i$, and so $A \lll B$.

Corollary 5. (Isbell) For a Hausdorff space X, the following are equivalent:

1. $O(X)$ is a continuous lattice.
2. X is locally compact.

Proof. Suppose that $O(X)$ is a continuous lattice, and let $x \in X$. Since $X = \sup \{A : A \ll X\}$, there is some $A \in O(X)$ with $x \in A \ll X$. Then, there is some $B \in O(X)$ with $A \ll B \ll X$, and it follows that $A \lll X$. This shows that $\overline{A}$ is compact by the Proposition, and so we have the desired compact neighborhood of x.

Conversely, suppose that X is locally compact, and let A be an open subset of X. Then, the local compactness and Hausdorff properties imply that A is the union of compact neighborhoods of each of the points in A, and the interior of such a neighborhood is then way-below A by the Proposition. Hence each open set of X is the sup of the open subsets way-below it, and so $O(X)$ is a continuous lattice.

Example 6. Let D be the closed unit disk in the plane, and let D have the usual topology. Let D' be the open unit disk. We define a new topology, k, on D as follows: A subset U of D is k-open if and only if, for each $x \in U$, there is an open subset V of D in the usual topology on D such that $x \in V$ and $V \cap D' \subseteq U \cap D'$. The effect of this is to give D' the usual topology, but the boundary of D is now discrete in the k-topology. We claim that D' is relatively compact in D in the k-topology: Indeed, let $\{O_i\}$ be a family of k-open sets which covers D. For each i, if $x \in O_i \cap D'$, we let $O_{i,x} = O_i$; if $x \in O_i \setminus D'$, then we let $O_{i,x} = (O_i \cap D') \cup V_x$, where $x \in V_x$ is an open subset of D in the usual topology such that $V_x \cap D' \subseteq O_i \cap D'$ (such a V_x exists by the definition of the k-topology). Now, since the family $\{O_i\}$ covers, it follows that the family $\{O_{i,x}\}$ covers D, and it is clear that each set $O_{i,x}$ is open in D in the usual topology. Hence, since D is compact in the usual topology, there is a finite subfamily $\{O_{j,x_j} : j = 1,\ldots,n\}$ which also covers D.

Now, $D' = D' \cap (\bigcup O_{j,x_j}) = \bigcup (D' \cap O_{j,x_j}) \subseteq \bigcup O_j$, since $O_{j,x_j} = (O_j \cap D') \cup V_{x_j}$ and $V_{x_j} \cap D' \subseteq O_j$ for each j. This shows that the family $O_1,\ldots,O_n$ forms a finite cover of D', and so we have our claim. It then follows that $D' \ll D$ in the k-topology.

The point of the example is to show that $U \ll V$ does not imply $\overline{U}$ compact even for Hausdorff spaces. The following result gives a characterization of $U \ll V$ for regular T_1 spaces.

Proposition 7. Let X be a regular T_1 space, and let Y be an open dense subset of X. The following are then equivalent:

1. $Y \ll X$ in $O(X)$.
2. Let $O'(X)$ be the family of all open sets U of X which satisfy: For each $x \in X$, if there is some $V \in O(X)$ with $x \in V$ and $V \cap Y \subseteq U \cap Y$, then $x \in U$. Then $O'(X)$ is a basis for a compact Hausdorff topology on X.

Note: The motivation for the topology $O'(X)$ given in part 2 stems from the idea of recovering the original topology on the unit disk D from the topology described in Example 5.

Proof. Suppose that 1 holds. It is routine to show that $O'(X)$ is a basis for a topology on X; moreover, if $x,y \in X$ with $x \neq y$, then there are disjoint open sets U and V containing x and y, respectively. Now, let $U' = \{z \in X : W \cap Y \subseteq U \cap Y$ for some open set W in X with $z \in W\}$, and let $V' = \{z \in X : W \cap Y \subseteq V \cap Y$ for some open set $W \subseteq X$ with $z \in W\}$. Then U' and V' are open in the new topology on X (they are infact members of $O'(X)$), and since U and V are disjoint, we have that U' and V' are disjoint. Moreover, clearly $U \subseteq U'$ and $V \subseteq V'$, so U' and V' are the disjoint open subsets in the new topology which we seek. Notice that a variant of this argument also shows that the new topology is regular, since the original topology is regular.

We now show that the new topology is compact. Let $\{A_i\}$ be a descending family of closed sets in the new topology, which, for brevity sake, we shall call the k-topology. Fix an index i, and let $x \in X$. If $x \notin A_i$, then since the k-topology is regular (as we noted above), there is a closed neighborhood $C(i,x)$ of A_i which doesn't contain x. If $x \in A_i$, then we let $C(i,x) = X$. For a finite subset F of X, we let $C(i,F) = \bigcap_{x \in F} C(i,x)$, which we note is a closed neighborhood of A_i. We claim the family $\{Y \cap C(i,F) : i \in I \text{ and } F \subseteq X \text{ is finite}\}$ has the finite intersection property. Indeed, suppose that $C(i_1,F_1),\ldots,C(i_n,F_n)$ are given. Then, $F = F_1 \cup \ldots \cup F_n$ is a finite subset of X, and since the family $\{A_i\}$ is descending, there is some A_j with $A_j \subseteq A_{i_k}$ for $k = 1,\ldots,n$. Then $C(j,F)$ is a closed neighborhood of A_j, as is $C(i_k,F_k)$ for each $k = 1,\ldots,n$, since $A_j \subseteq A_{i_k}$ for each $k = 1,\ldots,n$. Hence, since Y is dense in X, it follows that $Y \cap C(j,F) \cap \bigcap_{k \leq n} C(i_k,F_k) \neq \emptyset$, and this establishes the claim. Since Y is relatively compact in X in the original topology, it follows that $\bigcap \{C(i,F) : i \in I,\ F \subseteq X \text{ finite}\} \neq \emptyset$ since each of these sets has non-empty interior. Now, it is clear that $\bigcap A_i \subseteq \bigcap C(i,F)$; conversely, if $x \notin A_i$ for some i, then since $C(i,\{x\})$ is a closed neighborhood of A_i not containing x, it follows that $x \notin \bigcap C(i,F)$. Thus $\bigcap A_i = \bigcap C(i,F)$, and since the right side is non-empty, so also is the left. We have therefore shown that each descending family of closed subsets of X in the k-topology has a non-empty intersection, and so we conclude that X is compact in the k-topology. This finishes the proof that 1 implies 2.

Conversely, suppose that 2 holds, and let $\{O_i\}$ be an open cover of X in the original topology. For each index i, let $O_i' = \{z \in X : \text{there is an open set } V \text{ with } z \in V \text{ and } V \cap Y \subseteq O_i \cap Y\}$, and note that $O_i' \in O'(X)$ and $O_i \subseteq O_i'$ for each i. Hence, the family O_i' covers X, and since these sets are in $O'(X)$ which generates a compact topology, it follows that there is a finite subfamily $O_1',\ldots,O_n'$ which cover X. Now, $Y \cap (\bigcup_{j \leq n} O_j') = \bigcup_{j \leq n} (Y \cap O_j') \subseteq \bigcup_{j \leq n} O_j$ by the definition of O_j'. Hence, for the cover O_i of X, we have found a finite subcover $\{O_j : j \leq n\}$ which covers Y, and this shows that 1 holds.

This completes the proof of the Proposition.

The reason that this characterizes the way-below relation in $O(X)$ for regular T_1 spaces X is as follows: If $U \subset V$ are open sets, and if $U \ll V$, then we claim $U \ll \overline{U}$ in $O(\overline{U})$: Indeed, if $\{O_i\}$ is an open cover of $\overline{U}$, then each O_i can be written as $O_i' \cap \overline{U}$, where O_i' is open in X. Hence the family $\{O_i'\} \cup \{V \setminus \overline{U}\}$ is an open cover of V, and since $U \ll V$, there is a finite subcover of U. Clearly this gives rise to a finite subcover of U from the $\{O_i\}$. Conversely, suppose that $U \ll \overline{U}$ in $O(\overline{U})$. Then, for any open subset V of X with $\overline{U} \subset V$, it is easily seen that $U \ll V$ in $O(X)$. Thus, our Proposition does indeed characterize the way-below relation on $O(X)$ for X regular and T_1.

We now consider the question of whether $O(D)$ is a compact semialttice, where D is the unit disk with the topology described in Example 5. Since D is not locally compact in this topology, it is clear from Corollary 4 that $O(D)$ is not a continuous lattice. The following definition and lemma are taken from reference 2:

Definition 8. Let L be a complete lattice, and $A \subset L$ any subset. We define $A^+ = \{\sup B : B \subset A \text{ and } B \text{ is up-directed}\}$.

Lemma 9. (Lawson). Let S be a compact semilattice and I a semilattice ideal of S. Then, the closure of I satisfies $\overline{I} = I^{++}$.

Example 10. We show in this example that $O(D)$ is not a compact semilattice, where D is the unit disk with the topology described in Example 5. Let $E = D' \cup \{(1,0)\}$, and give E the relative topology from D. Then E is open in D, and so $O(E)$ is an ideal of $O(D)$ which is closed under up-directed sups. The lemma then implies that $O(E)$ is a compact semilattice if $O(D)$ is, and we show that this is not the case.

Let $x_0 = (1,0)$, and let $\{x_n\}_{n \geq 1}$ be a sequence in D' with no convergent subsequence in E (e.g., take $\{x_n\}$ to be a sequence which converges to $(0,1)$ in D in the usual topology). Define $A_n = E \setminus \{x_m\}_{m \geq n}$, and note that the family $\{A_n\}$ is an increasing set of open subsets of E whose union is all of E. Thus $O(A_n)$, the family of open subsets of A_n, is an ideal of $O(E)$, and the family $\{O(A_n)\}$ is an increasing family of ideals in $O(E)$. Now, let $\{y_n\}$ be a sequence in $D \setminus E$ which converges to x_0, and, for each n, choose a sequence $\{y_{n,m}\}$ in E which converges to y_n in the usual topology on D. We define $B_{n,m} = A_n \setminus \{y_{n,p} : p \geq m\}$, and note that, for each n, the family $\{B_{n,m}\}$ is an increasing family of open sets in E whose union is A_n; thus $O(B_{n,m})$ is an ideal of $O(E)$, and the family of these is up-directed. We let $I = \bigcup_{n,m} O(B_{n,m})$, and note that $E = \bigcup_n A_n = \bigcup_{n,m} B_{n,m} \in I^{++}$.

We claim that $E \notin I^{+}$: Indeed, suppose that $\{O_i\}$ is an up-directed subset of I and that $x_o \in \bigcup O_i$. Then, we can assume that $x_o \in O_i$ for each i. Fix i; since $\{y_n\}$ converges to x_o, and since $\{y_{n,m}\}$ converges to y_n for each n, it follows that there are r,s with $y_{p,q} \in O_i$ for $p \geq r$ and $q \geq s$. It then follows that $O_i \subset B_{n,m}$ implies $n \leq r$ and $m \leq s$, and so this also holds for each $j \geq i$ since the O_j are up-directed. Hence, $\bigcup O_j \subset \bigcup\{B_{n,m} : n \leq r \text{ and } m \leq s\} \subset A_r$, and so $\bigcup O_i \neq E$. This establishes our claim.

Now, we repeat the above construction as follows: We choose yet another sequence $\{z_n\} \subset D \setminus E$ which converges to x_o and which is disjoint from $\{y_n\}$, and for each n, we choose a sequence $\{z_{n,m}\}$ in E which converges to z_n in the usual topology of D. Now, for n,m in IN, we define $C_{n,m,p} = B_{n,m} \setminus \{z_{n+m,r} : r \geq p\}$, and note that $C_{n,m,p}$ is open on E for each triple n,m,p, and that the union of the $C_{n,m,p}$ for fixed n,m is $B_{n,m}$. This time we let $J = \bigcup_{n,m,p} O(C_{n,m,p})$, and note that J is an ideal of O(E). Now, $E = \bigcup_n A_n = \bigcup_{n,m} B_{n,m} = \bigcup_{n,m,p} C_{n,m,p} \in J^{+++}$.

We show that $E \notin J^{++}$: Suppose that $\{O_i\}$ is an up-directed subset of J^{+} and that $x_o \in \bigcup O_i$; then, as before we can assume x_o is in each O_i. We show that $O_i \subset B_{n,m}$ for some n,m depending on i; our previous remarks will then show that $E \neq \bigcup O_i$ since this is an up-directed family. Now, for a fixed i, $O_i \in J^{+}$, and so there is an up-directed family $\{O_{i,j}\}$ in J with $O_i = \bigcup O_{i,j}$. Again, we can assume that $x_o \in O_{i,j}$ for each j; fixing one j, since $O_{i,j}$ is open, there are r,s in IN with $z_{p,q} \in O_{i,j}$ for $p \geq r$ and $q \geq s$. It follows that $O_{i,j} \subset C_{n,m,p}$ for some n,m,p implies $n+m \leq r$ and $p \leq s$, so that $O_{i,j} \subset B_{n,m}$ and $n+m \leq r$. Since the $O_{i,j}$ are up-directed, it follows that $O_{i,k} \subset C_{n,m,p}$ implies $n+m \leq r$ and $p \leq s$ for $k \geq j$, and so $O_{i,k} \subset B_{n,m}$ with $n+m \leq r$ for $k \geq j$. We conclude that $O_i = \bigcup O_{i,j} \subset \bigcup_{n+m \leq r} B_{n,m} \subset B_{r,r}$, as is clear from the definitions. Thus each O_i is a subset of some $B_{n,m}$, and so they form an up-directed subset of I. Since $E \notin I^{+}$, it follows that $\bigcup O_i \neq E$. Therefore $E \notin J^{++}$.

Now, J is an ideal in the semilattice O(E) with $J^{++} \neq J^{+++}$. But, Lemma 8 shows that the closure of each ideal I is I^{++}, and clearly closed ideals are closed under the formation of up-directed sups (since these are then limits in the topology). Hence O(E) cannot be a compact semilattice, since J is an ideal with J^{++} not closed under up-directed sups, and so also not closed in any semilattice topology.

We close this memo with an observation on a result of Lawson's which appears in the proof of Theorem 13 of reference 2.

Definition 11. Let L be a complete lattice. For $x, y \in L$ we define $x \ll_c y$ if and only if for each subset A of L with $y \leq \sup A$, there is some countable subset $\{a_n\}$ of A with $x \leq \sup a_n$.

Lemma 12. For a compact semilattice S and an $x \in S$, we have $x = \sup \{y \in S : y \ll_c x\}$.

Proof. (Lawson(2)). For any compact neighborhood W_0 of x in S, choose recursively a family W_n of compact neighborhoods of x with $W_n^2 \subset W_{n-1}$ for each n. Let U be the intersection of the family W_n. Then, it is readily seen that U is a compact subsemilattice of S containing x. We let $u = \inf U$, and claim that $u \ll_c x$: Indeed, suppose that A is a subset of S with $x \leq \sup A$. Then, S is a compact semilattice, and so $x = \sup xA$. Hence, since $\sup A = \sup \{ \sup F : F \subset A \text{ finite}\}$, and the right side is an up-directed sup, the right side is also a limit. Hence, for each $n \in \mathbb{N}$, there is some finite subset $F_n \subset A$ with $\sup xF_n \in W_n$. Now, the set $C = \bigcup F_n$ is a countable subset of A, and $\sup xF_n \in W_n$ for each n implies that $\sup \{\sup xF_n : n \in \mathbb{N}\} = \lim \{\sup xF_n : n \in \mathbb{N}\} \in U$, and so $u = \inf U \leq \sup xC$. Since $\sup xC \leq \sup C$, it follows that $u \leq \sup C$, and we have established our claim.

Now, we have shown that, for each compact neighborhood W_0 of x, there is some $u \in W_0$ with $u \ll_c x$ (that $u \in W_0$ follows from the fact that $V \subset V^2$ for each subset V of S, so that the family W_n is towered). It then follows that $x = \sup \{y \in S : y \ll_c x\}$, and so the lemma is proved.

We note in closing the following properties of the Example 9: O(E) is a lower continuous complete lattice (since sups are just unions and finite infima just intersections), and O(E) satisfies the property that each $U \in O(E)$ is the sup of the $V \in O(E)$ with $V \ll_c U$ in O(E): this folows from the fact that E is a countable union of compact subsets, as is also each open subset of E. Finally, note that the proof that O(E) is not a compact semilattice can be used to show the following:

Proposition 13. Let X be a Hausdorff space which is embeddable in compact first countable (or, in particular, compact metrisable) space. Then the following are equivalent:

1. O(X) is a compact semilattice
2. O(X) is a continuous lattice.
3. X is locally compact.