SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC On the closedness of the set of primes in continuous lattices

REFERENCE
[1] K.Keimel and M.Mislove SCS-Memo 9-30-76, notably part 2

[2] K.H.Hofmann ,SCS-Memo 11-23-76

[3] O.Wyler, Algebraic theories of continuous lattices, preprint (old title Compact complete lattices).

[4] K.H.Hofmann and J.D. Lawson, Irreducibility..., preprint.

The following question is fairly pressing:

(P) Let S be a CL-object. When is $\overline{\text{PRIME } S}$ = PRIME S?

In [1] ,Keimel and Mislove have given a conclusive answer for distributive S, and in [4] the question of distributivity in CL was discussed at some length (Chapter 3). The condition Keimel and Mislove found to be necessary and sufficient in the distributive case was the following:

((0)) For all $a,x,y \in S$, the relations $a \ll x$ and $a \ll y$ imply $a \ll xy$.

This condition is evidently equivalent to any of the following

((0')) For any $s \in S$ the set int $\uparrow s$ is a filter. [2]

((0")) For all $a,b,x,y \in S$, the relations $a \ll x$ and $b \ll y$ imply $ab \ll xy$.

((0"')) Graph $\ll$ is a subsemilattice of $S \times S$.

Keimel and Mislove give an example of a sublattice of the square violating this condition.

O.Wyler proves the following fact in [4] 12.5, old version:

If L is a lattice with 0 and 1 , then PRIME $\hat{L}$ is closed , where $\hat{L}$ is the Z -dual of L (HMS -DUALITY).

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

The purpose of the Memo is to observe that this fact together with the observations by Keimel and Mislove in [1] suffice to show that ((0)) is sufficient for the closedness of PRIME S regardless of distributivity. In order to make this a bit more selfcontained, we present a proof of Wyler's proposition.

Recall that the CL -topology of a CL-object S has a basis of the open sets

$$U(u;v_1,\ldots,v_n) = \mathrm{int}\uparrow u \setminus (\uparrow v_1 \cup \ldots \cup \uparrow v_n),$$

$$\mathrm{int}\uparrow u = \{x \in S: u \ll x\}.$$

LEMMA 1 . If $v_1 \cdots v_n \leq u$, then $U(u;v_1,\ldots,v_n) \cap \mathrm{PRIME}\ S = \emptyset$.
Clear from the definitions.□

PROPOSITION 2. In a CL -object S, condition ((0)) implies $\overline{\mathrm{PRIME}}\ S = \mathrm{PRIME}\ S$.
Proof. Let $s \notin$ PRIME S. Then there exist elements $x,y \notin \downarrow s$ with $xy \leq s$. Since S is a continuous lattice, we find elements $a \ll x$ and $b \ll x$ with $a,b \notin \downarrow s$. By condition ((0")) we know $ab \ll xy$. Thus $U(ab;\ a,b)$ is an open neighborhood of s which according to Lemma 1 does not contain primes.□

RECALL 3 . An algebraic lattice $S \in Z$ is arithemetic if K(S) is a sublattice (i.e. is closed under finite infs).

REMARK 4. Any arithmetic lattice $S \in Z \subseteq CL$ satisfies condition ((0)).
Proof. Let $a \ll x,y$, i.e. $x,y \in \mathrm{int}\uparrow a$. Since $s = \sup(\downarrow s \cap K(S))$ for all $s \in S$, there are compact elements $c,k \in K(S)$ such that $a \ll c,k$ and $c \leq x$ and $k \leq y$. Since S is arithmetic, $ck \in K(S)$. Thus $a \leq ck \ll ck$,whence $a \ll ck$.□

steht auch in [9]

COROLLARY 5.(Wyler) The set of primes is closed in any arithmetic lattice.
This follows now from Proposition 2 and Remark 4.

COROLLARY 6. If $S \in CL$ and PS is as in ATLAS (the Z-object of all lattice ideals), then PRIME PS is closed.□

For distributive S this occurs in the proof of 2.1 in Keimel-Mislove [1].
From ATLAS we recall the morphism $r_S: PS \longrightarrow\!\!\!\!\to S$, $r_L(J) = \sup J$. By the preceding Corollary, r(PRIME PS) is a closed subspace of S which contains PRIME S (since $p \in$ PRIME S implies $\downarrow p \in$ PRIME PS). Keimel and

Mislove demonstrate that condition ((0)) implies that $J \in$ PRIME PS always gives sup $J \in$ PRIME S. We have

$$((n)) \Rightarrow ((n+1)):$$

((0))

((1)) If J is a prime ideal of S , then sup J is a prime.

((2)) $\overline{\text{PRIME}}$ S = PRIME S.

If S is distributive, then these conditions are equivalent. Example 4.2 in ATLAS satisfies ((2)) , but not ((0)).

Question. Does ((2)) imply ((1))?

If S satisfies ((2)) ,then the distributive CL-subobject

$S' = \{x \in S: \ x = \inf(\uparrow x \cap \text{PRIME } S)\}$ appears to play a role. It is not clear which. The closure operator $f: S \longrightarrow S'$, $f(s) = \inf(\uparrow s \cap \text{PRIME } S)$ is a lattice homomorphism. So what?