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# EQUATIONALLY COMPACT SENDOs ARE RETRACTS OF COMPACT ONES

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A SENDO is a semilattice $S, \wedge$ together with a semilattice endomorphism $x \mapsto \bar{x} : S \to S$.

S. BULMAN-FLEMING and I. FLEISCHER have shown: A SENDO is equationally compact iff the following properties hold:

(a) Every non-empty subset of $S$ has an infimum.

(b) Every $\uparrow$-directed subset of $S$ has a supremum.

(c) $a \wedge \bigvee d_i = \bigvee (a \wedge d_i)$ for every $a \in S$ and every $\uparrow$-directed subset $(d_i)$ in $S$.

(d) $\bigwedge \bar{x}_i = \overline{\bigwedge x_i}$ for every family $(x_i)$ in $S$.

(e) $\bigvee \bar{x}_i = \overline{\bigvee x_i}$ for every $\uparrow$-directed family $(x_i)$ in $S$.

They ask the question whether every equationally compact SENDO is the retract of a compact one. We now show that this question has a positive answer.

Let $S$ be a SENDO, which is equationally compact.

Let us suppose first, that $S$ has a greatest element $1$. Then (b) is a consequence of (a) and $S$ is a complete lattice.

Let $\mathcal{I}(S)$ be the lattice of all ideals of $S$. For every $I \in \mathcal{I}(S)$ let

$$\bar{I} = \{ x \in S \mid \exists i \in I : \ x \leq \bar{i} \}.$$

Then $\bar{I} \in \mathcal{I}(S)$. We have

(i) $\bigcup \bar{I}_j = \overline{\bigcup I_j}$ for every $\uparrow$-directed family $(I_j)$ in $\mathcal{I}(S)$.

(ii) $\bigcap \bar{I}_j = \overline{\bigcap I_j}$ for every non-empty family $(I_j)$ in $\mathcal{I}(S)$.

The proof of (i) is clear. In (ii) the inclusion $\supseteq$ is also evident. In order to show $\subseteq$ take $x \in \bigcap \bar{I}_j$. Then $x \leq \bar{i}_j$ for some $i_j \in I_j$ (all $j$). Then $x \leq \bigwedge \bar{i}_j$ (the infimum exist by $\boxed{(a)}$)

$$\bigwedge \bar{i}_j = \overline{\bigwedge i_j} \quad \text{by } \boxed{(d)}$$

As $\bigwedge i_j \in I_j$ for all $j$, we have $x \in \overline{\bigcap I_j}$, which ends the proof of (ii).

Now $\mathcal{I}(S)$ is an algebraic lattice. Endow $\mathcal{I}(S)$ with the topology that has as a subbasis the sets of the form $V(K) = \{ I \in \mathcal{I}(S) \mid K \subseteq I \}$ as

(7.11) As $x \prec y$ implies $x \ll y$, the map $I \mapsto \sup I : \mathcal{I}_{\sqsubset}(S) \to S$ is injective. We thus may identify $\mathcal{I}_{\sqsubset}(S)$ with its image in $S$. The map $x \mapsto \{y \in S \mid y \underset{S}{\prec^*} x\} \mapsto \sup \{y \in S \mid y \underset{S}{\prec^*} x\}$ will be denoted by $w_S : S \to S$.

(7.12) RESUME. Let $S$ be a compact semilattice. Let $w_S : S \to S$ be defined as above. Then $w_S$ is a kernel operator on $S$. Its range is a continuous lattice and $w_S : S \to w_S(S)$ is continuous, where $w_S(S)$ carries its compact semilattice topology (~~One can prove~~ not the induced topology). For every continuous lattice $T$ and every continuous homomorphism $g : S \to T$ there is a unique continuous homomorphism $g' : w_S(S) \to T$ such that $g' \circ w_S = g$.

(7.13) A direct proof of (7.12) can be given by taking word by word (6.7). One only has to replace $\ll$ by $\prec$ and to quote lemma (7.7) instead of (6.3).

Note that $w_S : S \to S$ factors through $v_S : S \to S$, i.e. the continous lattice $w_S(S)$ is a continuous homomorphic image of the continuous lattice $v_S(S)$. Problem (7.2) amounts to the question, whether $w_S = v_S$.