SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC	The spectral theory of distributive continuous lattices				

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SCS Keimel,Mislove 9-30-76
SCS Keimel,Mislove 12-15-76
SCS Hofmann,Wyler 12-28-76
SCS Hofmann, 1-13-77

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The spectral theory of lattices serves the purpose of representing a lattice L as a lattice of open sets of a topological space X. The spectral theory of rings and algebras practically reduces to this situation in view of the fact that for the most part one considers the lattice of ring (or algebra)ideals and then develops the spectral theory of that lattice.(The occasional complications due to the fact that ideal products are not intersections have been dealt with e.g. in THE RED BOOK.)

On the other hand, the question has now been raised repeatedly in the seminar and in the literature, what topological consequences would follow for a space X from the lattice theoretical assumption that the lattice O(X) of open sets was continuous. We have Isbell's observation that for Hausdorff X the local compactness of X is necessary and sufficient. SCS Keimel-Mislove 12-15-76 adresses itself further to this question,but reaches no conclusion in the absence of separation. We will show here that O(X) is a continuous lattice iff X is locally quasicompact -provided that every irreducibl (closed) subset of X is a singleton closure . More generally, we will show that the category of locally quasicompact T_0-spaces in whic all irreducible sets have a dense point with continuous maps as morphisms is dual to the category of distributive continuous lattices together with [illegible]-morphisms which are lattice morphisms, and SUP - [illegible]-morphisms.

Our main device is to use the hitherto somewhat neglected topology on a CL-object L which is generated by the sets $I(x)=L\setminus\uparrow x$ The l.u.b. of this topology and the Scott topology is the CL-topolog It induces on the set of primes precisely the hull-kernel topology. So it emerges that two T_0-topologies are of relevance on a continuous lattice. Until opposition from Oxford hits these shores we will call the one just introduced the anti-Scott-topology.

Der Worte sind genug gewechselt, laßt uns nun endlich Taten sehen! JWvG, F-1.

1. The basics.

1.1. NOTATION. Let L be a continuous lattice. We record the following topologies: (i) The Scott topology, generated by all $\Uparrow x = \{y \in L : x \ll y\}$, $x \in L$; (ii) The anti-Scott topology, generated by all $I(x) = L \setminus \uparrow x$, $x \in L$; and (iii) the CL-topology which is the common refinement of the Scott and anti-Scott topology. All of these topologies are T_0 and quasicompact, the last is T_2(and compact).

1.2. DEFINITION. Let L be a continuous lattice. We let Spec L be the space PRIME $L \setminus \{1\}$ with the topology induced from the anti -Scott topology and call this space the spectrum of L (or the prime spectrum, if confusion should ever arise). We notice that Spec L may be empty; if L is distributive, then PRIME L = IRR L order generates L (Hofmann, Lawson: Irreducibility) and Spec L is sufficiently large.

If $X \subseteq L$ we write $h(X) = \uparrow X \cap$ Spec L (and abbreviate $h(\{x\})$ by $h(x)$). Similarly we set $\sigma(X) =$(Spec L) $\setminus h(X) =$ (Spec L) $\setminus \uparrow X$, $\sigma(\{x\}) = \sigma(x)$. We call h(X) the hull of X. The topology of Spec L is generated by the $\sigma(x), x \in L$ and is called the hull-kernel topology. □

1.3. LEMMA . a) $\bigcap\{h(x) : x \in X\} = h(\sup X)$ for all $X \subseteq L$.

b) $\bigcup\{h(x) : x \in X\} = h(X) = h(\inf X)$ for all closed $X \subseteq L$.

c) Every hull-kernel closed set of Spec L is of the form $h(x)$ for some $x \in L$.

Proof. a) Is straightforward.

b) is immediate from THE LEMMA by Gierz and Keimel ("A lemma on primes" brings us all good times; see also Irreducibility 1.5).

c) The family $\{h(x) : x \in L\}$ is closed under arbitrary meets by a) and under finite unions by b). It therefore is the set of closed sets of a topology, which is the hull-kernel topology.[]

1.4. PROPOSITION. For any continuous lattice L, the function

$$x \longmapsto \sigma(x) : L \longrightarrow O(\text{Spec } L)$$

is a surjective lattice homomorphism preserving arbitrary sups. The following statements are equivalent:

(1) L is distributive. (2) $\sigma : L \longrightarrow O(\text{Spec } L)$ is an isomorphism.

Proof. The first assertion follows from Lemma 1.3. If L is distributive, then $\text{Spec } L \cup \{1\}$ is order generating, whence $x = \inf h(x)$. This means that σ is injective. Conversely, (2) says that σ is injective and hence that PRIME L is order generating which implies (1).[]

This proposition gives a representation of all continuous distributive lattice in the form $O(X)$. We have to understand the properties of the spaces which occur in this fashion.

1.5. LEMMA. If $F \subseteq L$ is an open filter, then $L \setminus F = {\downarrow}\sigma(F) = {\downarrow}((\text{Spec } L) \setminus$

Proof. Since F is a filter, $\sigma(F) = (\text{Spec } L) \setminus F$ by 1.2. If $x \leq p \in (\text{Spec } L) \setminus F$, then evidently $x \notin F$, i.e. $x \in L \setminus F$. Conversely, if $x \in L\setminus F$, then there is a $p \in \operatorname{Spec} L$ with $x \leq p$ and $p \notin F$ (Irreducibility 1.4), so $x \in \downarrow((\operatorname{Spec} L)\setminus F)$ []

1.6. LEMMA. A set $Q \subseteq \operatorname{Spec} L$ is quasicompact iff $\downarrow Q \subseteq L$ is compact in the CL-topology(or, equivalently, in the Scott topolc

Proof. A family $\{\sigma(a): A \subseteq L\}$ of open sets in Spec L is a cover of Q iff $Q \subseteq \bigcup\{\sigma(a): a \in A\} = \sigma(\sup A)$ iff $Q \setminus h(\sup A) = \emptyset$ iff $\sup A \notin \downarrow Q$. Thus Q has the Heine-Borel property iff for each set $A \subseteq L$ with $\sup A \notin \downarrow Q$ there is a finite subset $F \subseteq A$ with $\sup F \notin \downarrow Q$. This means precisely that $L \setminus \downarrow Q$ is open in the Scott topology. But upper sets are open in the Scott topology iff they are open in the CL-topology.[]

1.7. LEMMA. If $F \subseteq L$ is an open filter, then $\sigma(F) = (\operatorname{Spec} L)\setminus F$ is quasicompact in Spec L.

Proof. This is immediate from 1.5 and 1.6.[]

1.8. DEFINITION. A topological space X is called locally quasicompac iff every point has arbitrarily small quasicompact neighborhoods.[]

Note that in the absence of separation the existence of one quasi-compact neighborhood is not sufficient to guarantee local quasicompac ness.

1.9. LEMMA. Let X be a topological space.

(a) If $U,V \in O(X)$ and Q is quasicompact with $U \subseteq Q \subseteq V$, then $U \ll V$ in $O(X)$.

(b) If X is locally quasicompact, then $O(X)$ is a continuous latti [Day and Kelly]

Proof. (a) : Straightforward verification.

(b) : Immediate from the definition of continuous lattice, 1.8, and (a) above.[]

If $O(X)$ is a continuous lattice, then X is called semi-locally bound by Isbell (MC-lattices) and quasi-locally compact by A.S.Ward and Ω-compact by by B.J. Day and G.M.Kelly. [A.S.Ward in "Topology ar its applications, Blegrade 1969, p.352]

1.10. LEMMA . Let $a \ll b$ in L. Then there is a quasicompact set $Q \subseteq \text{Spec } L$ such that $\sigma(a) \subseteq Q \subseteq \sigma(b)$. Specifically, if F is any open filter of L with $b \in F \subseteq \uparrow a$, then $Q = \sigma(F)$ will do

Proof. There is indeed at least one open filter F with $b \in F \subseteq \uparrow a$ (since $a \ll b$ means $b \in \text{int} \uparrow a$, and thus $F = \uparrow U$ for any open semilattice neighborhood U of b in $\uparrow a$ will do). The relation $\sigma(a) \subseteq \sigma(F) \subseteq \sigma(b)$ is then clear, and $\sigma(F)$ is quasicompact by 1.7.▯

1.11. DEFINITION. A space X is called primal (Isbell) iff it is T_0 and every closed irreducible set has a dense point .(Here a closed set is called irreducible if is not the union of two proper non-empty closed subsets.)▯

Any infinite set with the cofinite topology is a non-primal T_1-space. Hausdorff spaces are primal.

1.12 .THEOREM . Let L be a continuous lattice. Then

(i) Spec L is a locally quasicompact T_0 space. In particular, O(Spec L) is a continuous lattice.

(ii) If L is distributive, then Spec L is primal.

(iii) The function $\sigma: L \longrightarrow O(\text{Spec } L)$ is a surjective CL^{op}-morphism. In particular, there is a CL -embedding

$\tau: O(\text{Spec } L) \longrightarrow L$ given by

$\tau(U) = \sup\{x \in L \mid \sigma(x) \subseteq U\} = \sup\{x \in L \mid h(x) \cup U = \text{Spec } L\}$.

Proof. (i) Since the anti-Scott topology is T_0, so is the hull-kernel topology on Spec L. In order to show that Spec L is locally quasicompact, let $p \in \sigma(x)$ for some $x \in L$. Pick a $y \ll x$ so that $p \notin \uparrow y$; this is possible since $p \notin h(x) = \uparrow x \cap \text{Spec } L$. Then $p \in \sigma(y)$, and by Lemma 1.10 there is a quasicompact Q with

$\sigma(y) \subseteq Q \subseteq \sigma(x)$. -By 1.9.b , O(Spec L) is a continuous lattice.

(ii) If L is distributive, then $\sigma : L \longrightarrow O(\text{Spec } L)$ is an isomorphism by 1.4. Let A be a non-empty closed irreducible set in Spec L. Then $A = h(a)$ for some $a \in L$ by 1.3.c. The set $\sigma(a) = (\text{Spec } L) \setminus h(a)$ is prime in O(Spec L) by irreducibility of A. Thus a is prime in L. Since $A \neq \emptyset$, then $a = \inf A \neq 1$, whence $a \in$ Spec L. But then $A = h(a) = \{a\}^-$ in Spec L. Thus Spec L is primal.

(iii) The function σ is a surjective lattice morphism preserving arbitrary sups by 1.4. If $x \ll y$ in L , then $\sigma(x) \ll \sigma(y)$ by 1.9.a and 1.10. Thus $\sigma \in \underline{CL}^{op}$. The remainder is clear from ATLAS duality.□

This theorem allows us to represent every distributive continuous lattice in the form O(X) for some locally quasicompact primal space X. This generalizes the representation theorem of Gierz and Keimel ("A Lemma on primes"). It also shows us how to find a canonical distributive subobject in any continuous lattice. (Cf. "Irreducibility", Chapter 3)

We now inspect the other direction: Starting from a space X, when do we recognize that O(X) is a continuous lattice?

Firstly, for every topological space X , O(X) is a complete Bouwerian lattice. We let Spec O(X) be the space of its primes in the hull-kernel topology which is the set $\{\sigma(U) : U \in O(X)\}$, where $\sigma(U) = \{ P \in \text{Spec } O(X) : U \nsubseteq P\}$.(For further information see e.g. THE RED BOOK, but be careful in comparing notation.)

1.13. <u>LEMMA</u>. Let X be a topological space and define the functi $\xi : X \longrightarrow \text{Spec } O(X)$ by $\xi(x) = X \setminus \{x\}^-$. Then ξ has the following properties:

(i) For all $U \in O(X)$ we have (a) $\xi(U) = \sigma(U) \cap \operatorname{im} \xi$ and
(b) $U = \xi^{-1}(\sigma(U))$.

(ii) $\sigma : O(X) \longrightarrow O(\operatorname{Spec} O(X))$ is a lattice isomorphism with inverse $V \longmapsto \xi^{-1}(V)$.

(iii) ξ is continuous and open onto its image, and Spec $O(X)$ is a primal space.

(iv) ξ is injective iff ξ is an embedding iff X is T_o.

(v) ξ is bijective iff ξ is a homeomorphism iff X is primal.

Proof. (i)(a) An open set $P \in \operatorname{Spec} O(X)$ is in $\sigma(U)$ iff $U \not\subseteq P$; hence $X \setminus \{x\}^-$ is in $\sigma(U) \cap \operatorname{im} \xi$ iff $U \not\subseteq X \setminus \{x\}^-$ iff $\{x\}^- \cap U \neq \emptyset$ iff $x \in U$ iff $X \setminus \{x\}^- \in \xi(U)$.

(b) An element $x \in X$ is in $\xi^{-1}(\sigma(U))$ iff $\xi(x) \in \sigma(U)$ iff $U \not\subseteq X \setminus \{x\}^-$ iff $x \in U$.

(ii) is a consequence of (i) (b) and the fact that σ is surjective.

(iii): The first part follows from (i) (b) and (a), respectively, and the second from (ii) as in the proof of 1.12 (ii).

(iv) and (v) are immediate from the definitions in view of (iii).

1.14 . <u>DEFINITION</u>. If $j : X \longrightarrow Y$ is an embedding of topological spaces then we call j <u>strict</u> if $U \longmapsto j^{-1}(U) : O(Y) \longrightarrow O(X)$ is an isomorphism of lattices.

Notice that for T_o-spaces X the map ξ is a strict embedding by 1.13.

1.15. <u>LEMMA</u>. Let L be a continuous lattice and $X \subseteq \operatorname{Spec} L$. Then the following statements are equivalent:

(1) The inclusion $X \longrightarrow$ Spec L is a strict embedding (relative to the hull kernel topology on X)

(2) $X \cup \{1\}$ is order generating in L .

Remark. In "Irreducibility" 2.2 one finds for alternative equivalent conditions for condition (2).

Proof. Condition (1) means that for all s,t $\in$ L the relation $\sigma(s) \cap X = \sigma(t) \cap X$ implies $s=t$. This is equivalent to

(1') For all $s,t \in L$, the relation $\uparrow s \cap X = \uparrow t \cap X$ implies $s = t$.

Since $\uparrow s \cap X = \uparrow t \cap X$ is equivalent to $\uparrow s \cap (X \cup \{1\}) = \uparrow t \cap (X \cup \{1\})$ we note that "Irreducibility" 2.2 shows that (1') and (2) are equivalen

These concepts are particulalry easily applied to the case of algebraic lattices . For this purpose let L be an algebraic lattice ($L \in \underline{Z}$) and let $X \subseteq$ Spec L be a strictly embedded subspace. By 1.15 and "Irreducibility" 2.5 , this implies Irr $L \subseteq X \cup \{1\}$. We then confirm parallels to 1.5,1.6 and 1.10 as follows:

1.5.bis.<u>LEMMA</u> . If $F \subseteq L$ is an open closed filter, then $L \setminus F = \downarrow(X \setminus F)$.
Proof. We need only confirm $L \setminus F \subseteq \downarrow(X \setminus F)$: Let $s \in L \setminus F$; then by "Irreducibility 1.4 there is a $p \in$ Irr L with $s \leq p$ and $p \notin F$. Since Irr $L \subseteq X \cup \{1\}$,we have $p \in X$.[]

For $A \subseteq L$ let us write $\sigma_X(A) = X \setminus \uparrow A = \sigma(A) \cap X$.

1.6.bis. <u>LEMMA</u>. A set $Q \subseteq X$ is hull-kernel quasicompact iff $\downarrow Q$ is closed in L (relative to the <u>CL</u>-topology).
Proof. The proof of Lemma 1.6 applies with σ_X in place of σ .[]

1.10.bis. <u>LEMMA</u>. Let $a \ll b$ in L/ Then there is a quasicompact open set Q such that $\sigma_X(a) \subseteq Q \subseteq \sigma_X(b)$. Specifically, if F is an open-closed filter of L with $b \in F \subseteq \uparrow a$, then $Q = \sigma_X(F)$ will do.

Proof.Mimic the proof of1.10 with σ_X in place of σ and with an open closed filter in place of an open filter.[]

1.12.bis. THEOREM. Let L be an algebraic lattice. Then

(i bis) every strictly embedded subspace $X \subseteq$ Spec L is a T_0-space with a basis of quasicompact open sets. In particular, $O(X)$ is an algebraic lattice.

Proof. The proof of 1.12 (i) adapts with the aid of Lemma 1.10.bis.□

We now summarize:

1.16. THEOREM. For a T_0-space X the following statements are equivalent:

(1) $O(X)$ is a continuous lattice.

(2) [resp. (2')] X allows a strict dense embedding into a locally quasicompact [primal] space.

(3) There is a continuous distributive lattice L such that X may be considered as a subspace of Spec L in such a fashion that $X \cup \{1\}$ is order generating in L.

Furthermore, the following statements are equivalent:

(I) $O(X)$ is an algebraic lattice.

(II) X has a basis of quasicompact open sets.

(III) X allows a strict dense embedding into a primal space with a basis of quasicompact open sets.

(IV) There is a distributive algebraic lattice L such that X may be considered as a subspace of Spec L with $\mathrm{Irr}\, L \setminus \{1\} \subseteq X$.

Proof. (3)=>(2'): By 1.12, Spec L is a locally quasicompact primal space. Thus (3) implies (2') by 1.15 . (2')=>(2) is trivial. (2)=>(1) follows from 1.9.b and 1.14. (1)=>(3) : Let $L = O(X)$. Then ξ:X—>Spec L is a strict embedding by 1.13 and 1.14. Then $\xi(X) \cup \{1\}$ is order - generating in L by 1.15.

(I) <=>(II) is immediate from the definitions, and in view of 1.15, Theorem 1.12 bis does (IV)=>(I). Next (II)=>(III): Consider the

strict embedding $\xi: X \longrightarrow \text{Spec}\,(O(X))$ into a primal space by 1.13. Then $O(X) = O(\text{Spec}(O(X)))$ by 1.13 ii . Thus Spec $O(X)$ has a basis of quasicompact open sets since $O(\text{Spec}\ O(X))$ is algebraic.
(III) => (IV): (III) =>(2') <=> (3) and since $L = O(X)$ we know that L is algebraic; then the conclusion $\text{Irr}\ L \setminus \{1\} \subseteq X$ follows from "Irreducibility" 2.5.□

1.17 THEOREM . Let X be a primal space. Then the~~x~~ following conditions are equivalent:

(1) $O(X)$ is continuous lattice. (2) X is locally quasicompact.

Moreover, if these conditions are satisfied, then $U \ll V$ in $O(X)$ iff there is a quasicompact ~~xx~~ $Q \subseteq X$ with $U \subseteq Q \subseteq V$.

Remark. If X has a basis of quasicompact sets, then $U \ll V$ iff there is a quasicompact Q with $U \subseteq Q \subseteq V$, as is immediately verified.

Proof. (2) => (1): 1.~~7~~9.b.

(1) => (2): By 1.14.(v) , $\xi: X \longrightarrow \text{Spec}\ O(X)$ is a homeomorphism. By 1.12, Spec $O(X)$ is locally quasicompact.

If $U \subseteq Q \subseteq V$ for a quasicompact Q, then always $U \ll V$ (cf.1.9.a). Conversely, ~~if~~ let $U \ll V$. We recall that we may identify X with Spec L for some continuous distributive lattice L as soon as X is primal and locally quasicompact. In that case, Lemma 1.10 yields the required Q.□

1.18. COROLLARY (Isbell). For a Hausdorff space X the lattice $O(X)$ is continuous iff X is locally compact.□

Theorem 1.16 characterizes T_0-spaces X for which $O(X)$ is compact provided one understands the concept of strict dense subspaces of locally quasicompact primal spaces or, alternatively, order generating subsets of PRIME L for continuous distributive lattices L.

As far as primal spaces are concerned, they are in bijective correspondence with distributive continuous lattices by 1.12 and 1.17.

We make the following observation which ties in with the duality theory presented by Lawson in SCS-memo 1-4-77.

1.22. PROPOSITION. Let L be a continuous lattice. The function $\sigma : (\mathcal{OF}, \cap) \longrightarrow (\mathcal{QS}, \cup)$, $\sigma(F) = \mathrm{Spec}\, L \setminus F$ from the $\cap$-semilattice of open filters of L into the $\cup$-semilattice of quasicompact saturated sets is a surjective semilattice morphism and is in fact an isomorphism if L is distributive.

Remark. $(\mathcal{OF}, \cap)$ is the dual of L in the sense of Lawson (SCS memo 1-4-77).

Proof. It is clear that σ is a semilattice homomorphism, and by 1.21 it is surjective. If L is distributive, whence PRIME L is order generating, and so different open filters have different hulls hence different σ-images.$\square$

We turn to a purely topological concept.

1.23. DEFINITION . Let X be a topological space and 1 and element with $1 \notin X$. The patch topology on $X \cup \{1\}$ is the topology generated by $O(X)$ and the collection of all $(X \setminus Q) \cup \{1\}$ where Q is a quasicompact saturated subset of X.$\square$

1.19. <u>NOTATION</u> . Let X be a topological space. For $x, y \in X$ we write $x \le y$ iff $y \in \{x\}^-$. This is a transitive relation and a partial order if X is T_0 . The set $\downarrow Y$ is called the <u>saturation</u> of $Y \subseteq X$, and Y is <u>saturated</u> iff $\downarrow Y = Y$.▯

<u>Note</u>. If $L \in \underline{CL}$, then the partial order induced by that of L on Spec L agrees with the one given on Spec L by 1.19.

The following observations should be clear:

1.20 <u>REMARK</u> . All open sets of a space are saturated. The saturation of a set Y is the intersection of all open sets containing Y. The set Y is saturated iff Y is an intersection of open sets. The saturation of a quasicompact set is quasicompact. A space is locally quasicompact iff every point has arbitrarily small saturated quasicompact neighborhoo

1.21. <u>LEMMA</u>. Let L be a continuous lattice, and $Q \subseteq$ Spec L. Consider the following conditions:

(1) Q is closed in L (relative to the $\underline{CL}$-topology).

(2) Q is quasicompact in Spec L.

(3) Q is quasicompact saturated in Spec L.

(4) There is an open filter F in L such that $Q = \text{Spec } L \setminus F$.

Then (1)=>(2) <=(3)<=>(4). If PRIME L is closed and Q is saturated in Spec L , then (1) -(4) are equivalent.

Proof. (1)=>(2): If Q is closed in L then so is $LQ = \downarrow Q$ since L is a compact topological semilattice. Thus Q is quasicompact in Spec L by 1.6. (3)=>(2) is trivial. (4)=>(3) follows from 1.7. (3)=>(4): By 1.20 and (3), Q is the intersection of open sets in Spec , whence $Q = \text{Spec } L \setminus F$ where $F = \{x \in L \mid \uparrow x \cap Q = \emptyset\}$. Since $Q \subseteq$ PRIME L , then F is a filter. Now let $x \in F$. Then $\{\sigma(y) : y \ll x\}$ is an open cover of Q. Since Q is quasicompact by (3), there is a $y \ll x$ with $Q \cap \uparrow y = \emptyset$. Thus for each $x \in F$ there is a $y \ll x$ with $\uparrow y \in F$, and thus F is open. Finally suppose that Q is saturated (whence (2)<=>(3)) and that PRIME L is closed. Suppose (2). Then $\downarrow Q$ is closed in L by 1.6. Thus $\overline{Q} \subseteq \downarrow Q \cap (\text{PRIME}$ $= \downarrow Q \cap$ PRIME L , and the last intersection is Q since Q is saturated in Spec L. ▯

1.24a LEMMA If L is a continuous lattice, then the patch topology on Spec L $\cup$ {1} is coarser than or equal to the topology of PRIME L (induced from the CL-topology). It is always Hausdorff.

Proof. The "new" closed sets in the patch topology are of the form $Q = \downarrow Q \cap$ Spec L with $\downarrow Q$ closed in the CL-topology, whence the first assertion. Now suppose $p \neq q$ in Spec L $\cup$ {1}. Suppose that $p \notin \downarrow q$. Then take a point $x \ll q$ in L such thzt $p \notin \downarrow x$ and an open filter $F \subseteq \uparrow x$ with $q \in F$. Then $\sigma(F) =$ Spec L $\setminus$ F is a saturated quasicompact set not containing q. Then $\sigma(x)$ and h(F) are disjoint neighborhoods of p, respectively q in the patch topology.□

1.24b. LEMMA . The patch topology on Spec L $\cup$ {1} is compact iff PRIME L is closed in L.

Proof. Suppose Spec L $\cup${1} is compact is the patch topology. By 1.24.a this topology agrees with that of PRIME L which is, therefore, compact. So PRIME L is closed in L. If PRIME L is closed, then a saturat set $Q \subseteq$ Spec L is hull-kernel quasicompact iff it is closed in CL by 1.21. The "new" closed sets in the patch topology are simply the intersections with PRIME L of all closed lower sets. But these together with the intersection with PRIME L of all closed upper sets generate the closed sets of the CL-topology on PRIME L.□

1.25. THEOREM. Let L be a distributive continuous lattice and X = Spec L .[Note that X is a locally quasicompact primal space and that every such space occurs precisely in this fashion.] Then the following statements are equivalent:

((0)) For all $x,a,b \in L$, the relations $x \ll a,b$ imply $x \ll ab$.

(1) PRIME L is closed in L.

(2) The collection of saturated quasicompact sets in X is closed under (finite) intersections.

(3) The patch topology on $X \cup \{1\}$ is compact.

Proof. ((0)) => (1) : SCS mem Hofmann, Wyler.

(1) => (2): By 1.21 a saturated set $Q \subseteq X$ is quasicompact iff Q is closed in L in the CL-topology. A finite collection of CL-closed sets has a CL -closed intersection (and the intersection of any collection of saturated sets is saturated).

(2) => ((0)): Let $x \ll a,b$. Then $\sigma(x) \ll \sigma(a), \sigma(b)$ by 1.12.iii. By 1.17 there are quasicompact saturated subsets P,Q in X with $\sigma(x) \subseteq P \subseteq \sigma(a)$ and $\sigma(x) \subseteq Q \subseteq \sigma(b)$. By (2) $P \cap Q$ is quasicompact, and $\sigma(x) \subseteq P \cap Q \subseteq \sigma(a) \cap \sigma(b) = \sigma(ab)$. Then $x \ll ab$ by 1.17 and 1.12 .

(1) <=> (3) : Lemma 1.24.□

1.26. ZUSATZ. Under the hypotheses of 1.25, the conditions ((0))-(3) are also equivalent to the following

(4) For every prime ideal $I \subseteq L$ we have sup $I \in$ PRIME L.

Proof. See SCS Keimel-Mislove 9-30-76 and SCS Hofmann - Wyler.□

1.27. ZUSATZ. If L is an arithmetic lattice (i.e. an algebraic lattice such that K(L) is a sublattice), and if X = Spec L, then the equivalent conditions ((0)) - (4) in 1.25 and 1.26 are satisfied.

Proof. SCS Hofmann-Wyler.□

1.28. <u>COROLLARY</u>. Let V (Verband) be an arbitrary distributive lattice and L = PV the lattice of all lattice ideals (including ∅ if V has no smallest element). Let X = Spec L = set of all prime ideals of V with the hull kernel topology. Then conditions ((0))-(4) in 1.25 and 1.26 are satisfied.

Proof. We know that L is an algebraic lattice with $K(L) = \{\uparrow v : v \in V\} \cup \{0_L\}$. Hence L is arithmetic. BY THE YELLOW BOOK we know that L is distributive iff K(L) is distributive. Hence L is distributive, and 1.27 applies.□

REMARK. In comparing THE RED BOOK with what is done here one should notice that THE RED BOOK calls Spec V what we here would have to call Spec PV. THE RED BOOK uses prime <u>ideals</u> (equivalently, characters) as basic ingredient, we use prime <u>elements</u>. The transition between the two is guaranteed by the functor P on which ATLAS says a lot.

The patch topology is extensively used in the spectral theory of commutative rings. (Hochster, Grothendieck.)

1.29. <u>PROPOSITION.</u> (Gierz-Keimel) Let L be a distributive continuous lattice in which the equivalent conditions of Theorem 1.25 aresatisfied. Then L is isomorphic to the lattice of open saturated sets in the patch topology of Spec L ∪ {1}.

Proof. By Theorem 1.25 and Lemma 1.23 the patch topology on Spec L ∪{1} is the <u>CL</u>-topology on PRIME L and PRIME L is CL-closed. Then the hull-kernel closed sets of PRIME L are precisely the <u>CL</u>-closed upper sets of PRIME L, i.e. the sets $\sigma(x)$, $x \in L$ and PRIME L are precisely the patch -open lower sets. The assertion then follows from 1.4.□

2. The duality between distributive continuous lattices and locally quasicompact primal spaces

We complement the considerations of Section 1 by taking the morphisms into account. The present observations are somewhat in the spirit of the RED BOOK.

We need some notation which pinpoints our morphisms.

2.1. DEFINITION. Let $\underline{CL}(\wedge, \bigvee)$ the category of all continuous lattices and lattice morphisms preserving arbitrary sups. Let $\underline{CTop}$ be the category of all topological spaces X such that O(X) is a continuous lattice and all continuous maps.

2.2. LEMMA . Let $f: L \longrightarrow S$ be in $\underline{CL}(\wedge, \bigvee)$, and let $g: S \longrightarrow L$ be its left adjoint. Then $g(\text{Spec } S) \subseteq \text{Spec } L$, and the restriction and corestriction of g defines a continuous function Spec f: Spec S ——> Spec T.

Proof. Since g preserves infs, $g(1) = g(\inf \emptyset) = \inf g(\emptyset) = 1$. If $p \in$ PRIME S, then $S \setminus {\downarrow}p$ is a filter, so $L \setminus f^{-1}({\downarrow}p) = f^{-1}(S \setminus {\downarrow}p)$ is a filter since f preserves finite infs. Since f preserves sups, and g is a left adjoint, we have $g(p) = \max f^{-1}({\downarrow}p)$ (see ATLAS), whence ${\downarrow}g(p) = f^{-1}({\downarrow}p)$. Thus $L \setminus {\downarrow}g(s)$ is a filter, whence g(s) is prime. Thus $g(\text{Spec } S) \subseteq \text{Spec } L$. Furthermore, for $x \in L$ we observe $g^{-1}(h(x)) = \{p \in \text{Spec } S \mid g(p) \geq x\} = \{p \in \text{Spec } S \mid p \geq f(x)\} = h(f(x))$, since g is left adjoint to f. Thus g is hull-kernel continuous. $\square$

Recall that a map between topological spaces is proper if the invers images of quasicompact sets are quasicompact.

We will say that a map is decent , if the inverse images of saturated quasicompact sets are quasicompact.

2.3. LEMMA. If, under the circumstances of Lemma 2.2. the map f is in addition a $\underline{CL}^{op}$-morphism, then Spec(f) : Spec S ---> Spec L is decent.

Proof. Let Q be a saturated quasicompact set in Spec L. Then $Q = \text{Spec } L \setminus F$ for some open filter F in L by 1.21. Then $(\text{Spec } f)^{-1}(Q) = g^{-1}(\text{Spec } L \setminus F) \cap \text{Spec } S = \text{Spec } S \setminus g^{-1}(F)$. Since $f \in \underline{CL}^{op}$, then $g \in \underline{CL}$ and so $g^{-1}(F)$ is an open filter. By 1.7, we know then that $(\text{Spec } f)^{-1}(Q)$ is quasicompact. □

2.4. LEMMA If f: X--->Y is in CTop , then $O(f):O(Y) \longrightarrow O(X)$ given by $O(f)(V) = f^{-1}(V)$ is in $\underline{CL}(\wedge, \bigvee)$.

Proof. Clear.

2.5. LEMMA. If in addition to the hypotheses of 2.4, the spaces X and Y are primal and f is decent, then O(f) is in $\underline{CL}^{op}$.

Proof. Let $U \ll V$ in O(Y). Then there is a saturated quasicompact set Q with $U \subseteq Q \subseteq V$ (1.17 and 1.20). Then $O(f)(U) \subseteq f^{-1}(Q) \subseteq O(f)(V)$ and $f^{-1}(Q)$ is quasicompact since f is decent. Then $O(f)(U) \ll O(f)(V)$ by 1.17. □

We now add to the umpteen adjunction theorems in the RED BOOK another one:

2.6. PROPOSITION . The assignments Spec : $CL(\wedge, \bigvee)$ ---> CTop and O: CTop ---> $CL(\wedge, \bigvee)$ are contravariant functors which are adjoint on the right (i.e. Spec: $CL(\wedge, \bigvee)$ ---> $\underline{CTop}^{op}$ is left adjoint to O: $\underline{CTop}^{op}$ ---> $CL(\wedge, \bigvee)$). The adjunctions are $\sigma_L : L \longrightarrow O(\text{Spec } L)$ and $\xi_X : X \longrightarrow \text{Spec } O(X)$. The adjunction σ_L is an isomorphism iff L is distributive and the adjunction ξ_X is a homeomorphism iff X is primal locally quasicompac[t]. The functor O o Spec : $CL(\wedge, \bigvee)$ ---> $CL(\wedge, \bigvee)$ is an epi-reflector onto the full subcategory of distributive continuous lattices, and the functor Spec o O: CTop ---> CTop is an epi-reflector onto the full subcategory of primal locally quasi-compact spaces.

Proof. Spec and O clearly are contravariant functors. The adjunction follows from THE FIFTH ADJUNCTION THEOREM 4.3 of the RED BOOK (p.39) and may also be verified directly. The assertions on the adjunctions come from 1.4 and 1.13 in conjunction with 1.17. The remainder is standard general nonsense. []

2.7. THEOREM. The cztegory $\underline{CL}_{dist}(\wedge, \bigvee)$ of distributive continuous lattices with lattice homomorphisms preserving arbitrary sups and the category $\underline{LQCP}$ of locally quasicompact primal spaces and continuous maps are dual under Spec and O. Under this duality, the subcategory $\underline{CL}_{dist}(\wedge, \bigvee) \cap \underline{CL}^{op}$ corresponds to the subcategory $\underline{LQCP}_{dec}$ of locally quasicompact primal spaces and decent continuous maps.

This Theorem is contained in the FIRST DUALITY THEOREM 4.17 on p.46 of the RED BOOK. It adds another case to the SECOND DUALITY THEOREM 5.6 on p.50 of the RED BOOK , and this case generalizes the duality between $C_2 = \underline{Z}$ and the category K_2 (= full subcategory of LQCP of spaces having a basis of quasicompact open sets). See also Proposition 1.42 on p.73 of the YELLOW BOOK.