SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC	Complement to " The spectral theory of distributive continuous lattices				
REFERENCE	Hofmann-Lawson SCS 2-8-77				

Let us call a space X with O(X) ∈ CL a CL-space; this is a preliminary nomenclature until agreement can be reached as to what these spaces ought to be called. So far we have : A.S.Ward: ~~xix~~ Quasi locally compact spaces; J.Isbell: Semilocally bounded spaces.

In essence, our two memos (op.cit and this one) are concerned with a characterisation of CL-paces. We recall a general fact (loc. cit (1.13):

FACT I: For every T_0 space X there is a unique universal "primalisation" $\xi_X : X \longrightarrow \check{X}$, where $\check{X}$ is a primal(loc.cit.1.11) space and ξ_X is a dense embedding inducing an isomorphism $O(\check{X}) \cong O(X)$

"Universal" means that every map X--->Y into a primal space factors uniquely through ξ_X; we did not emphasize this fact in SCS 2-8-77. Recall that $\check{X}$ = Spec O(X).

We have the following result on CL- spaces:

FACT II (op cit 1.16) For a T_0 space X the following statements are equivalent:

(1) X is a CL- space. (2) $\check{X}$ is locally quasicompact.

(3) X ~~is~~ may be identified with a subset of Spec L for a continuous distributive CL-object L so that X∪{1} is order generating.

The following question remained open: (~~a~~ first occurrence known to us: A.S.Ward, Proceedings of the Herceg Novi Topology conference in 1968): Is each CL-space necessarily locally quasicompact?

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

MIT (Isbell)
Carnegie Mellon (Wyler)
Queen's U.(Giles)

The answer is no as we now proceed to communicate. There is in fact a second countable CL-space in which every quasicompact subspace has empty interior. The lattice of its open ideals is the following: For Scott : $[I \longrightarrow I]$; For Isbell $\mathcal{F}n_{\Omega}(I,I)$; for Hofmann $LC(I,I)$. This example will show that the classification theorem FACT 2 is pretty much the best possible.

Let now $L = LC(I,I)$, $I = [0,1]$, LC denotes the classically lower semicontinuous functions. For any $(a,b) \in I \times [0,1[$ let $p_{(a,b)} \in L$ be the lower semicontinuous function given by $p_{(a,b)}(a) = b$ and $= 1$ otherwise. We note that L is distributive and that Spec L (loc.cit.1.2) $= \{p_{(a,b)} : (a,b) \in I \times [0,1[\}$ If we equip $Y = [0,1] \times [0,1[$ with the topology consisting of all $\{(x,y) \mid y < f(x)\}$, $f \in LC(I,I)$, then $(a,b) \longmapsto p_{(a,b)} : Y \longrightarrow$ Sepc L is a homeomorphism. Notice that Y is second countable.

We define $X \subseteq Y$ as follows. The axiom of choice enables us to fix a subset $A \subseteq I$ with the following properties: (1) A is dense in I . ~~(2) I\A is dense in I~~ (2) A ∩U is not Borel for any ope U≠∅ in I. (We could have gotten A ~~not~~ nowhere Lebesgue measurable in I).
We say $(x,y) \in X$ iff $y \in]0,1[$ is rational for $x \in A$ and irrational in $]0,1[$ for $x \in I \setminus A$. If X' is the image of X in Spec L , then $X' \cup \{1\}$ clearly order generates all of Spec L and thus all of L. Hence X is a CL-space by FACT 2.

In order to show that each quasicompact subset of X has empty interior it suffices to show that every saturated quasicompact subset (loc.cit.1,19-1.20) has empty interior. Thus let Q be a saturated quasicompact subset of X. Saturation means that $(a,b) \in Q$ implies $\{a\} \times [0,b] \subseteq Q$. ~~Since A × {0} is a quasicompact subset (without interior) it is no loss of generality to assume that A × {0} ⊆ Q from here on out.~~

Lemma A. $q(x) = \max \{y \mid (x,y) \in Q\}$ exists for all $x \in pr_1 Q$.
Proof. The collection $Q \cap (\{x\} \times [s - \frac{1}{n}, s]), n=1,2,\ldots$, where $s = \sup \{y \mid (x,y) \in Q\}$ is a filterbasis of closed subsets of the quasicompact space Q and thus has a non-empty intersection in Q. But the only point in this intersection is (x,s).$\square$

Define $q: I \longrightarrow I$ by $q(x) = 0$ for $x \notin pr_1 Q$, and as in Lemma A, otherwise.

Lemma B. $q: I \longrightarrow I$ is upper semicontinuous.

Proof. Let $x = \lim x_n$ in I and suppose that (x,y) is a limit point of $(x_n, q(x_n))$ in the standard topology of $I \times \mathbb{R}$. ~~The topology of Q is coarser than the standard topology. Hence $(x,y) = \lim (x, q(x_n))$ in Q.~~ By the definition of the topology on Y, the relation $(x,y) = \lim (x_n, q(x_n))$ in the standard topology implies that for any cluster point (x,z) (in Y) of the sequence $(x_n, q(x_n))$ we have $y \leq z$. By the quasicompactness of $Q \cup (I \times \{0\})$, at least one of these cluster points is in $Q \cup (I \times \{0\})$. Thus $y \leq q(x)$ by the definition of q. □

Lemma C. If $b: I \longrightarrow I$ is a Borel function, then $b^{-1}(Q^+)$ is a Borel subset of I, where Q^+ denotes the set of positive rationals.

Clear, since Q^+ is Borel. □

Now q is a Borel function since it is upper semicontinuous by B above.

Lemma D. $pr_1 Q \cap A$ is a Borel subset of A.

Proof. By the definition of A and X we have $pr_1 Q \cap A = q^{-1}(Q^+)$. □

If Q had a non-empty interior, then $pr_1 Q$ would contain a non-empty open subset U, whence $A \cap U$ would be a Borel set contrary to the selection of A.

QED.

Flash:

Al Stralka has a characterisation of all CL-objects which are quotients of products of chains (at least of those which are first countable at the identity).