SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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Topic: Quotients of Distributive Continuous Lattices: A result of S.A. Jalali

Refs: All the references of "Spectral Theory" by Hofmann / Lawson

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Dr. S. A. Jalali of the Mathematical Insitute, Oxford, has made last week what I think is a very interesting discovery about projections of topological lattices. <u>We should be very glad to be informed whether the result is known</u>.

In my terminology a projection is just a <u>kernel</u> operator on a continuous lattice, the image of which is just a <u>quotient</u> in the category CL with the usual $\sqcap$-preserving, $\sqcup\uparrow$-continuous maps. We recall from ATLAS, that if L and M are continuous lattices, then for L to be a quotient of M, we only need find a function $i: L \to M$ with

(i) i is one-one;

(ii) i preserves all $\sqcup$'s;

(iii) i preserves $\ll$.

We call the projection <u>faithful</u> if in addition:

(iv) $i(\top_L) = \top_M$.

JALALI'S THEOREM. (i) Every continuous lattice L is the quotient of the lattice $\mathcal{O}(L)$ of open subsets of L in the Hausdorff (Lawson) topology.

(ii) If X and Y are compact Hausdorff spaces and the continuous map $f: X \to Y$ is onto, then $\mathcal{O}(Y)$ is a quotient of $\mathcal{O}(X)$ and the projection is faithful.

(iii) In (i), if $\top$ is isolated in L, then L is a faithful projection of $\mathcal{O}(L - \{\top\})$.

Jalali then remarks that it is a classical theorem of Alexandroff and Urysohn that every compact Hausdorff space with a countable basis is a continuous image of the Cantor space $2^{\mathbb{N}}$. His theorem then shows directly that every continuous lattice with a countable basis is a faithful projection of $\mathcal{O}(2^{\mathbb{N}})$. In my more constructive proof, I had to first obtain a projection onto an algebraic lattice and then only retract onto a continuous lattice. No, it comes to the same thing since a retract is a quotient of a subalgebra. In any case, this theorem explains what is going on in very clear topological terms. Jalali also points out that it also generalizes to suitable higher cardinals for spaces of uncountable weight. I shall, however, not give the details here.

Proof of (i). We define $i : L \to \mathcal{O}(L)$ by

$$i(a) = \{x \in L \mid a \not\equiv x\}$$

Properties (i) and (ii) are clear. For (iii) we must show that $a \ll b$ implies $cl(i(a)) \subseteq i(b)$. This means:

$$x \in cl(i(a)) \to x \in i(b)\ ;$$

that is:

$$\forall \mathcal{U}\,[x \in \mathcal{U} \to \exists y \in \mathcal{U}\,.\, a \not\equiv y\,] \to b \not\equiv x\,.$$

Equivalently:

$$b \equiv x \to \exists \mathcal{U}\,[x \in \mathcal{U} \wedge \forall y \in \mathcal{U}\,.\, a \equiv y\,].$$

Since we assume $a \ll b$, we need only take

$$\mathcal{U} = \{z \mid a \ll z\}.$$

Proof of (iii). In case T is isolated in L, we have $\{T\}$ open. Also $L \sim \{T\}$ is a compact Hausdorff space, and the map

$$i(a) = \{x \in L \sim \{T\} \mid a \not\equiv x\}$$

is faithful.

Proof of (ii). In the case of the two spaces X and Y, we obviously put:

$$i(\mathcal{U}) = f^{-1}(\mathcal{U}).$$

Since f is continuous and onto it satisfies all the conditions including faithfulness. $\square$