SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC: COMMENTS ON THE SPECTRAL THEORY OF CONTINUOUS LATTICES

REFERENCES: [1] K.H. Hofmann and J.D. Lawson, The spectral theory of distributive continuous lattices. SCS Memo, 2/8/77.

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Theorem 1.12 of [1] says in part too much, in part too little. The proof of parts (i) and (iii) is based on 1.7 which in turn is based on 1.5. The given proof of 1.5 is only valid for the distributive case, when every irreducible element is prime. On the other hand, 1.12 (ii) is valid for every complete lattice.

In this memo, I define Spec as a contravariant functor from complete lattices, with Scott continuous lattice homomorphisms, to topological spaces, and I use the adjunction of Spec and $\underline{0}$ to obtain the duality between sober (= primal) topological spaces and complete lattices with enough primes. This part is known. Specializing to continuous lattices, I show that (i) and (iii) in Thm. 1.2 of [1] are both valid iff Lemma 1.7 of [1] is valid. This is the case if L is distributive, and also if $\operatorname{Spec} L \cup \{1\}$ is closed in the CL-topology of L. Thus L need not be distributive for (i) and (iii) in Thm. 1.12 of [1] to hold.

The problem whether 1.7 in [1] holds for all continuous lattices remains open.

1. Enough primes versus sobriety

Topologists have used spaces of closed sets for over sixty years, beginning with the Hausdorff metric. Both from a topological viewpoint and from the look of the covariant functors involved, spaces of closed sets are preferable compared with spaces of open sets. However, in order to preserve the notations of [1], I shall use here complete lattices and spaces of open sets.

We deal with two categories. One is the category TOP of topological spaces and continuous maps. On the other side is the category of complete lattices and Scott continuous lattice homomorphisms. I denote this category by LSC . Isbell's meet-continuous lattices, and Dowker's frames (which are meet-continuous and distributive) define full subcategories of LSC . The morphisms of LSC preserve finite meets (including 1) and arbitrary suprema.

Every morphism $f : L \longrightarrow M$ of LSC has a right adjoint $f^* : M \longrightarrow L$, given by $x \leqslant f^*(y) \iff f(x) \leqslant y$; one may be tempted to call f^* a geometric morphism of complete lattices. f^* preserves arbitrary infima and primes. Conversely, if $f^* : M \longrightarrow L$ preserves infima and primes, and if M has enough primes (see below), then the corresponding $f : L \longrightarrow M$ preserves finite meets as well as all suprema.

We define a (contravariant) functor $\underline{O} : \mathrm{TOP}^{op} \longrightarrow \mathrm{LSC}$ in the usual way: $\underline{O}X$ is the complete lattice of open sets of X for a space X , and $\underline{O}f : \underline{O}Y \longrightarrow \underline{O}X$ is given by inverse images for $f : X \longrightarrow Y$.

In the other direction, $\mathrm{Spec} : \mathrm{LSC}^{op} \longrightarrow \mathrm{TOP}$ is defined as follows. Spec L is the set of all primes of L , with 1 excluded, and with the sets $\sigma_L(a) = \{p \in \mathrm{Spec}\, L \mid a \nleq p\}$ as open sets. For $f : L \longrightarrow M$ in LSC , we have noted that the right adjoint f^* preserves primes; we denote by $\mathrm{Spec}\, f : \mathrm{Spec}\, M \longrightarrow \mathrm{Spec}\, L$ the restriction of f^* to primes. Then $(\mathrm{Spec}\, f)^{-1}(\sigma_L(a)) = \sigma_L(f(a))$ for $a \in L$; thus Spec f is continuous.

One verifies easily that $\sigma_L : L \longrightarrow \underline{O}\,\mathrm{Spec}\, L$ is a surjective morphism of LSC ; the preceding paragraph shows that σ_L is natural in L .

THEOREM 1. <u>The functors</u> $\underline{O} : \mathrm{TOP}^{op} \longrightarrow \mathrm{LSC}$ <u>and</u> $\mathrm{Spec} : \mathrm{LSC}^{op} \longrightarrow \mathrm{TOP}$ <u>are adjoint on the right</u>.

<u>Proof</u>. For $u : X \longrightarrow \mathrm{Spec}\, L$ in TOP, put $\hat{u} = \underline{O}u \cdot \sigma_L$; then $\hat{u} : L \longrightarrow \underline{O}X$ in LSC. For $v : L \longrightarrow \underline{O}X$ in LSC and $x \in X$, put

$$\hat{v}(x) = \sup\{a \in L \mid x \notin g(a)\} .$$

Then $v(\hat{v}(x)) = \bigcup\{g(a) \mid x \notin g(a)\}$. It follows that

$$a \leq \hat{v}(x) \iff x \notin v(a) \iff v(a) \subset X \setminus \overline{\{x\}}$$

Thus $\hat{v}(x) = v^*(X \setminus \overline{\{x\}})$. This is prime in L since $X \setminus \overline{\{x\}}$ is prime in $\underline{O}X$, and we have $\hat{v} : X \longrightarrow \mathrm{Spec}\, L$ at the set level. $\hat{v}^{-1}(\sigma_L(a)) = v(a)$ from the displayed equivalence. Thus $\hat{v} : X \longrightarrow \mathrm{Spec}\, L$ in TOP, and $\hat{\hat{v}} = \underline{O}\hat{v} \cdot \sigma_L = v$.

If $v = \hat{u}$, and $x \in X$, we have in particular

$$\begin{aligned} \hat{\hat{u}}(x) &= \sup\{a \in L \mid x \notin u^{-1}(\sigma_L(a))\} \\ &= \sup\{a \in L \mid a \leq u(x)\} = u(x) . \end{aligned}$$

Thus $\hat{\hat{u}} = u$.

Now $u \longmapsto \hat{u}$ and $v \longmapsto \hat{v}$ defines inverse bijections which are clearly natural in X and in L respectively; this proves the Theorem.

One unit of the contravariant adjunction just obtained is $\sigma_L = (\mathrm{id\ Spec}\, L)^\wedge$. We denote the other unit by $s_X = (\mathrm{id}\ \underline{O}\, X)^\wedge : X \longrightarrow \mathrm{Spec}\ \underline{O}\, X$. The elements of $\mathrm{Spec}\ \underline{O}\, X$ are the complements of the irreducible closed sets of X, and we have in particular $s_X(x) = X \setminus \overline{\{x\}}$ for $x \in X$.

PROPOSITION. $\mathrm{Spec}\, \sigma_L$ <u>and</u> $s_{\mathrm{Spec}\, L}$ <u>are inverse homeomorphisms for a complete lattice</u> L, <u>and</u> $\underline{O}\, s_X$ <u>and</u> $\sigma_{\underline{O}X}$ <u>are inverse isomorphisms for a topological space</u> X.

<u>Proof</u>. We have $\mathrm{Spec}\, \sigma_L \cdot s_{\mathrm{Spec}\, L} = \mathrm{id\ Spec}\, L$ and $\underline{O}\, s_X \cdot \sigma_{\underline{O}X} = \mathrm{id}\ \underline{O}\, X$ since σ and s are the units of a contravariant adjunction. Since σ_L is epimorphic, $\mathrm{Spec}\, \sigma_L$ is monomorphic; the first part follows. $\sigma_{\underline{O}X}$ is epimorphic; this implies the second part.

THEOREM 2. <u>Sober topological spaces define a full reflective subcategory of</u> TOP, <u>with reflections</u> $s_X : X \longrightarrow \mathrm{Spec}\ \underline{O}\, X$. <u>For a topological space</u> X, <u>the following four statements are logically equivalent</u>. (i) X <u>is sober</u>. (ii) s_X <u>is bijective</u>. (iii) s_X <u>is a homeomorphism</u>. (iv) X <u>is homeomorphic to a space</u> $\mathrm{Spec}\, L$, <u>for a complete lattice</u> L.

Proof. (i) $\Longrightarrow$ (ii) by definition of sobriety. (ii) $\Longrightarrow$ (iii) since $\underline{O}\, s_X$ is bijective in any case. (iii) $\Longrightarrow$ (iv) trivially, and (iv) $\Longrightarrow$ (iii) by the preceding Proposition.

If $f : X \longrightarrow Y$ with Y sober, then $f = g\, s_X$ only if $\underline{O}\, g = (\underline{O}\, s_X)^{-1}\, \underline{O}f$ As $\underline{O}$ is full and faithful for sober spaces, this determines g uniquely []

DEFINITION. We say that a complete lattice L has enough primes if σ_L is injective, and hence bijective.

THEOREM 3. Complete lattices with enough primes define a full reflective subcategory of LSC, with reflections $\sigma_L : L \longrightarrow \underline{O}\,\mathrm{Spec}\, L$. For a complete lattice L, the following three statements are logically equivalent. (i) L has enough primes. (ii) σ_L is an isomorphism of complete lattices. (iii) L is isomorphic to a complete lattice $\underline{O}\, X$, for a topological space X.

Proof. (i) $\Longrightarrow$ (ii) since every bijective morphism of LSC is an isomorphism of complete lattices. (ii) $\Longrightarrow$ (iii) trivially. (iii) $\Longrightarrow$ (i) since $\underline{O}\, X$ with primes $X \setminus \overline{\{x\}}$ has enough primes.

If $f : L \longrightarrow M$ in LSC with M having enough primes, then $f = g\sigma_L$ for at most one g since σ_L is epimorphic, and $f = g\sigma_L$ for $g = \sigma_M^{-1} \circ \underline{O}\,\mathrm{Spec}\, f$.

Theorems 2 and 3 are a special case of a folk-theorem written up recently by Lambek and Rattray. The two theorems are not new, but they have not been written down in the form best suited to spectral theory of CL.

2. Spectral theory of continuous lattices

THEOREM 4. For a continuous lattice L, the following two statements are logically equivalent. (i) $\mathrm{Spec}\, L$ is locally compact, and $\sigma_L : L \longrightarrow \underline{O}\,\mathrm{Spec}\, L$ is a comorphism of continuous lattices. (ii) For every open filter F in L, the set $\mathrm{Spec}\, L \setminus F$ is compact in $\mathrm{Spec}\, L$.

Before proving this Theorem, we state and prove an auxiliary result.

PROPOSITION. (i) and (ii) in Theorem 4 are satisfied if L is distributive, and also if $\mathrm{Spec}\, L \cup \{1\}$ is closed in the CL-topology of L.

Proof. If L is distributive, then (ii) in Thm. 4 is valid by 1.7 in [1]. If $\operatorname{Spec} L \cup \{1\}$ is CL-closed and $Q = \operatorname{Spec} L \setminus F$ for an open filter F in L, then Q is CL-closed. But then $\downarrow Q$ is CL-closed in L, and Q is compact in Spec L by Lemma 1.6 of [1].

Proof of Theorem 4. The proof of parts (i) and (iii) of Thm. 1.12 in [1] is a proof of (i) from (ii).

If L satisfies (i) and $X = \operatorname{Spec} L$, then $\sigma_L^* : \underline{O}X \to L$ is an embedding in $\underline{CL}$. If F is an open filter in L and $G = (\sigma_L^*)^{-1}(F)$, then G is an open filter in $\underline{O}X$, and $\operatorname{Spec} \underline{O}X \setminus G$ is compact since (ii) is valid for the distributive continuous lattice $\underline{O}X$. Now $X \setminus s_X^{-1}(G)$ is compact in X since s_X is a homeomorphism. $s_X(p) = \sigma_L(p)$ for $p \in X$; thus $s_X^{-1}(G)$ consists of all $p \in X$ with $\sigma_L(p)$ in G, i.e. with $\sigma_L^*(\sigma_L(p))$ in F. We always have $\sigma_L^*(\sigma_L(a)) \geq a$, but since $\sigma_L \sigma_L^* \sigma_L = \sigma_L$ and $p \notin \sigma_L(p)$, we cannot have $\sigma_L^*(\sigma_L(p)) > p$. Thus $s_X^{-1}(G) = X \cap F$, and $X \setminus F$ is compact in X.

REMARK. As pointed out in [1], there are distributive continuous lattices L such that $\operatorname{Spec} L \cup \{1\}$ is not CL-closed in L. On the other hand, if L is the complete lattice of filters in a lattice ℓ, i.e. the HMS dual of ℓ, then Spec L with the Stone space topology is a closed subspace of L with the CL topology, but L is distributive only if ℓ is distributive.

We note that Lemma 1.24a of [1] can be sharpened.

PROPOSITION. *If* L *is a continuous lattice which satisfies the equivalent conditions (i) and (ii) of Theorem 4, then the patch topology of* $\operatorname{Spec} L \cup \{1\}$ *is the subspace topology induced by the* CL *topology of* L.

Proof. Put $X = \operatorname{Spec} L$. We have injective mappings $X \cup \{1\} \to \underline{O}X \to L$, given by s_X, extended by $s_X(1) = X$, and σ_L^*. As the morphism σ_L^* of $\underline{CL}$ is a closed embedding for the CL topologies, we assert that the patch topology is the initial topology induced by s_X. This follows immediately from the fact that the CL topology of $\underline{O}X$, for locally compact X, is the coarsest topology for which $\uparrow V$ is closed for every $V \in \underline{O}X$, and the filter $\uparrow Q$ of open neighborhoods of Q open for every compact $Q \subset X$.