SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC

REFERENCE

QUOTIENTS OF CUBES

In this note we will investigate conditions under which a compact semilattice is the quotient of a product of compact chains. Of necessity such a semilattice must be an object in $C\mathcal{L}$, the category of compact Lawson semilattices. We shall also discuss conditions under which a compact (necessarily zero-dimensional) semilattice is a quotient of 2^P for some set P. Definitions of undefinied terms are to be found in [4].

Let $\{T_j : j \in J\}$ be a family of compact (non-degenerate) chains and let T be their cartesian product. Define $T_j' = \{t \in T : pv_i(t) = 1$ unless $i = j\}$ ($pv_j(t)$ is the j^{th} projection map). IRR(S) will denote the set of (meet) irreducible elements of S. Thus in T, $\mathrm{IRR}(T) = \bigcup \{T_j' : j \in J\}$.

LEMMA 1 (Hofmann and Lawson [2]): Let $\varphi : A \to B$ be a $C\mathcal{L}$-surmorphism, then $\mathrm{IRR}(B) \subseteq \varphi(\mathrm{IRR}(A))$.

If S belongs to $C\mathcal{L}$ and $x, y \in S$, then $x \ll y$, read x is way below y, if whenever $\sup A \geq y$ there is a finite subset F of A such that $\sup F \geq x$. This condition is equivalent to: y is in the interior of $\uparrow x$. ($\uparrow x = \{s \in S : s \geq x\}$. The set $\downarrow x$ is defined dually).

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PROPOSITION 2: Let $S \in C\mathcal{L}$. The following conditions are equivalent.

(1) S is a quotient of a product of compact (non-degenerate) chains.

(2) $IRR(S)$ can be written as a union of a family of chains $\{D_j : j \in J\}$ in such a way that if $x \ll 1$ then there is a finite subset F of J such that if $j \in J/F$ then $D_j \subseteq \uparrow x$.

PROOF: (1) $\Rightarrow$ (2). Suppose that $q:T \to S$ is a quotient map (T is as described above). For each $j \in J$, define D_j to be $q(T_j') \cap IRR(S)$. By Lemma 1, $IRR(S) = \cup\{D_j : j \in J\}$. Let $x \ll 1$ in S. Then $Q^{-1}(\uparrow x)$ is a neighborhood of 1 in T. Hence there is a finite subset F of J such that for each $j \in F$, U_j is a neighborhood of 1 in T_j and $\cap\{pr_j^{-1}(U_j) : j \in F\} \subseteq q^{-1}(\uparrow x)$. Since $\cup\{T_i' : i \neq j\} \subseteq pr_j^{-1}(U_j)$ for each $j \in F$ we have $\cup\{T_j' : j \in F\} \subseteq q^{-1}(\uparrow x)$ which implies that $D_j \subseteq \uparrow x$ for all $j \in -J\backslash F$.

(2) $\Rightarrow$ (1). Define $T_j = \overline{D_j} \cup \{1\}$. Thus $T = \Pi\{T_j : j \in J\}$ is a product of compact chains. Define $q:T \to S$ by setting $q((t_j)_{j \in J}) = \inf\{t_j : j \in J\}$. By (2.19[2]) q preserves arbitrary infs and is a surjection. We have left to show that ir preserves sups of upward directed sets. Let $(t_j^{\alpha})_{j \in J}$ be an upward directed set in T which converges to $(t_j)_{j \in J}$. Let $x \ll q((t_j)_{j \in J}) = \inf\{t_j : j \in J\}$. By condition

(2) we can find the appropriate finite set F . Then $q((t_j^{\alpha})j \in J) = (\inf\{t_j^{\alpha}:j \in F\}) \wedge (\inf\{t_j^{\alpha}:j \in J\backslash F\}) \geqq x \wedge (\inf\{t_j^{\alpha}:j \in F\})$ which converges to $x \wedge (\inf\{t_j:j \in F\}) \geqq x \wedge (\inf\{t_j:j \in J\}) = x \wedge Q((t_j)_{j \in J}) \geqq x$. Then since this set is upward directed we can conclude that $\lim q((t_j^{\alpha})_{j \in J}) = q((t_j)_{j \in J})$. □

Recall that a set has finite width if it does not contain an infinite anti-chain.

LEMMA 3: Let $S \in C\mathcal{L}$. The following two conditions are equivalent.

(P_1) given any infinite anti-chain A in $IRR(S)$, $\overline{A} = A \cup \{1\}$.

(Q_1) if $x << 1$ in S then $IRR(S)\backslash \uparrow x$ has finite width.

PROOF: $(P_1) \Rightarrow (Q_1)$. Suppose that $x << 1$ in S . If $IRR(S)\backslash \uparrow x$ did not have finite width it would have an infinite anti-chain A . But then A could not have 1 in its closure since $\uparrow x$ is a neighborhood of 1 .

$(Q_1) \Rightarrow (P_1)$. Let A be an infinite anti-chain in $IRR(S)$ and let b be a limit point of A . Each neighborhood of 1 contains all but finitely many members of A . Thus $\overline{A} = A \cup \{1\}$. □

A stronger condition than (P_1) is

(P_2): given any infinite subset A of $IRR(S)$, $\overline{A} = A \cup \{1\}$.

PROPOSITION 4: If $S \in C\mathcal{L}$ is a quotient of a product of compact chains then S has property (P_1) and if S is a quotient of 2^P, for some set P, then S has property (P_2).

PROOF: Suppose that $T = \Pi\{T_j : j \in J\}$ is a product of compact (non-degenerate) chains. We may assume that J is infinite: otherwise, P_1 is vacuously satisfied. Let A be any infinite anit-chain in T and let b be a limit point of A. For each $j \in J$ there is at most one element of A whose $j^{\underline{th}}$ projection is different from 1. Thus for each $j \in J$, there is a neighborhood U_j of b such that $pr_j(A \cap U_j) = 1$. Hence $b = 1$.

Now suppose that $q : T \to S$ is a quotient map and suppose that B is an infinite anti-chain in $IRR(S)$, By Lemma 1, for each $b \in B$ we can find a point $b' \in q^{-1}(b) \cap IRR(T)$. $B' = \{b' : b \in B\}$ will be an infinite anti-chain in $IRR(T)$. Hence, from the previous paragraph $\overline{B}' = \overline{B}' \cup \{1\}$. Thus $q(\overline{B}') = B \cup \{1\}$. Since q is continuous we have $\overline{B} = B \cup \{1\}$.

The second part of our proposition follows from the fact that infinite subsets of $IRR(2^P)$ become anti-chains upon the exclusion of 1. □

We are only able to supply a partial converse to Proposition 4. The condition that there be a countable neighborhood base at 1 would not appear to be a necessary hypothesis. Nevertheless, we have found no way to eliminate it as a hypothesis.

PROPOSITION 5: Let $S \in C\mathcal{L}$ and suppose that S has a countable neighborhood base at 1. Then S is a quotient of a product of compact chains if and only if S satisfies property (P_1).

PROOF: From Proposition 4 we have the result in one direction. Now suppose that S satisfies property P_1. From the hypothesis of a countable neighborhood base at 1, we can find a sequence $\{e_i : i = 1,2,\ldots\}$ such that for each i, $e_i \ll e_{i+1}$ and $\sup e_i = 1$. Since $IRR(S)\setminus\uparrow e_1$ has finite width it can be expressed as a union of finitely many chains by Dilworth's coding theorem, call these chains $D_n,\ldots,D_{1n(1)}$. $IRR(S)\setminus\uparrow e_1 \subseteq IRR(S)\setminus\uparrow e_2$ so each D_{1i} can be extended to a maximal chain D_{2i} in $IRR(S)\setminus\uparrow e_2$. The remaining elements of $IRR(S)\setminus\uparrow e_2$ has finite width so it can be arranged into finitely many chains. Thus we have $IRR(S)\setminus\uparrow e_2 = D_{21} \cup \ldots \cup D_{2n(1)} \cup \ldots \cup D_{2n(2)}$. As this process proceeds it uses up $IRR(S)$. Hence by Proposition 2 we see that S is a quotient of a product of chains. □

For the 2^P case we have

PROPOSITION 6: Let $S \in C\mathcal{L}$ and suppose that 1 has a countable neighborhood base. Then S is a quotient of 2^P, for some set P, if and only if S satisfies P_2.

PROOF: The proof in one direction is taken care of in Proposition 4. If S satisfies condition P_2 we can proceed along on the same course as Proposition 5. However, since IRR(S) would have only one limit point in this case, each of the chains we obtain would have to be either finite or countably infinite and having one as its sole limit point. Hence if we let T_i denote the countable compact chain with one as its sole limit point we can see that T^N (N is the set of natural numbers) has S as a quotient. But then since T is a quotient of 2^N it follows that T^N must also be a quotient of 2^N we have completed our proof. □

In [1] Baker considered the question of which semilattices can be imbedded into free semilattices. If $j:A \to F(S)$ is an imbedding of the semilattice A into the free semilattice on the set S. Then by the duality theory of [3] there would be a surmorphism $\hat{j}:F\hat{(S)} \to \hat{A}$. The semilattice $F\hat{(S)}$ is isomorphic with 2^S. Thus the question of which semilattices can be imbedded into free semilattices is dual to the question of which semilattices are quotients of products of finite chains.

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