Oberwolfach 30 March 1976

Dear Hofmann:

I am spending four weeks here at the Forschungsinstitut for a little peace and quiet. Already I have quite by chance uncovered many things in the excellent library I would otherwise never have known about. That is one of the special pleasures of being here.

Thank you very much for your very helpful letter of last month. The business of the term was such that I had no time to answer before this.

I have gone over most of the ATLAS and I think I understand Galois connections now. Also I understand the Hausdorff topology. It is, by the way, very easy to see that it is compact. As you point out we have as a subbase sets of the forms:

$$\{x \mid a \ll x\} \qquad \{x \mid b \not\sqsubseteq x\}$$

(Sorry! I mix your notation with mine.) Now, by a standard argument using prime ideals of sets, we have the covering property for open sets iff we have it for the subbase. So suppose a family of sets like those above, determined by a_i $(i \in I)$ and b_j $(j \in J)$, covers, but that no finite subfamily does. Thus for any $i \in I$ and any finite $J_0 \subseteq J$ there is an x where $a_i \not\ll x$ and $b_j \sqsubseteq x$, all $j \in J_0$. Therefore

$$a_i \not\ll \bigsqcup_{j \in J_0} b_j \quad \text{because} \quad \bigsqcup_{j \in J_0} b_j \sqsubseteq x .$$

But consider $\bigsqcup_{j \in J} b_j$. This poor element must belong somewhere, but the only kind of open set into which it could fit would give $a \ll \bigsqcup_{j \in J} b_j$ for some $i \in I$. The properties of $\ll$ lead directly to a contradiction.

Note that the argument just given applies to lattices without a unit. I always write $\bot \sqsubseteq x \sqsubseteq \top$ to avoid the arithmetic notation $0 \leq x \leq 1$ which gets confused with so many other things. But the top element $\top$ is a great nuisance for many applications. You, of course, will find that statement odd, since you think of the semigroup $\langle S, \sqcap, \top \rangle$ with unit. But perhaps there is something to interest you in leaving off the unit. Axiomatically it is easy to explain the generalization of continuous lattices as follows:

(i) $\langle S, \sqsubseteq \rangle$ is a poset;

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(ii) Every bounded set has a lub;

(iii) Every directed set (or chain) has a lub;

(iv) Every element is the lub of elements $\ll$ it.

If we add a $\top$ or change (ii) to every set, we are back to the old theory. In the argument for compactness, the lub $\bigsqcup\{b_j \mid j \in J_0\}$ exists because it is bounded; therefore, the big lub $\bigsqcup\{b_j \mid j \in J\}$ exists because it comes from a directed set. So, on S, we have two topologies: my favourite "from below" T_0-topology and for you a compact Hausdorff topology.

I would guess that the kind of morphisms you want are those $g: S \to T$ which are (1) continuous from below (what you call "normal" in a quite unnecessary definition 1.29 and is my usual notion of continuous) and (2) provided with an adjoint $d: T \to S$ (that is to say: have at least one d). Now we can see at once that g is Hausdorff continuous because:

$$g^{-1}\{x \in T \mid b \not\sqsupseteq x\} = \{y \in S \mid b \not\sqsupseteq g(y)\}$$
$$= \{y \in S \mid d(b) \not\sqsupseteq y\},$$

and this last is always open in S. Since $g^{-1}\{x \in T \mid a \ll x\}$ is assumed open, we find that g is continuous in the sense we want. Also g preserves glb's of all non-empty sets. We can no doubt replace (2) by that condition plus saying $g(S)$ is cofinal in T; that is,

$$\forall x \in T\ \exists y \in S.\ x \sqsubseteq g(y).$$

That makes the definition of d meaningful:

$$d(x) = \sqcap\{y \in S \mid x \sqsubseteq g(y)\}.$$

(Without cofinality the duality does not seem to go through.)

Perhaps you will not particularly like this generalization, but I only mention it because it has come up so often and so often the $\top$ seems very artificial. Here are some models without $\top$:

(1) Take any S in $\mathcal{CL}$ and any subset $\mathcal{U} \subseteq S$ that is open "from below"; that is, open in my sense or equivalently a union of sets $\{x \mid a \ll x\}$. If $S \setminus \mathcal{U}$ is non-empty then this closed set is just such a "hemilattice" (what should we call them?) — and what is more, every such can be obtained in this way from a $S \in \mathcal{CL}$ where $\mathcal{U} = \{\top\}$ is open.

(2) For example, consider "interval arithmetic" where S consists of intervals $[\underline{a}, \bar{a}]$ with $\underline{a} \leq \bar{a}$ and

either $-\infty < \underline{a} \leq \bar{a} < +\infty$ or $-\infty = \underline{a} \leq \bar{a} = +\infty$.

Clearly $\perp = [-\infty, +\infty]$ and we have

$$a \sqsubseteq b \text{ iff } \underline{b} \leq \underline{a} \leq \bar{a} \leq \bar{b}$$

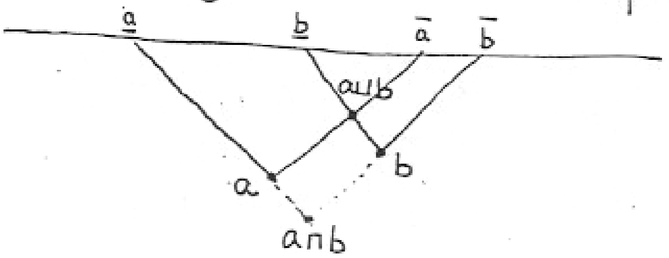
$$a \sqcap b = [\min(\underline{a}, \underline{b}), \max(\bar{a}, \bar{b})]$$

$$a \sqcup b = [\max(\underline{a}, \underline{b}), \min(\bar{a}, \bar{b})] \text{ \underline{if non empty}}$$

$$a + b = [\underline{a} + \underline{b}, \bar{a} + \bar{b}]$$

$$-a = [-\bar{a}, -\underline{a}] \quad \text{etc. etc.}$$

(See Springer Lecture Notes in Computer Science vol 29 "Interval Mathematics" ed. by K. Nickel (Karlsruhe) for more information on possible practical applications.) If you want you can make this a lattice by taking $\top = \emptyset =$ the empty "interval", but some people do not want to. Pictorially we can think of the lower half plane

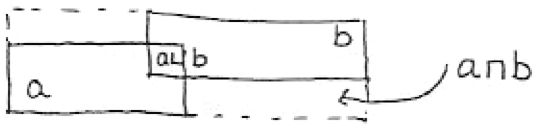


where each pt represents an interval by drawing lines at 45°. In this way we get an ordinary lattice diagram as shown. I guess that the Hausdorff topology is necessarily the usual one but $\perp$ gives a one-point compactification.

I am sure you are familiar with these examples, and we can go to higher dimensions. Thus use all points in 3-space on or below the x,y-plane. "Intervals" now become 45° right triangles. Looking from the top we see:

Given two such triangles a and b, then $a \sqcap b$ is the least containing both. As sets (closed triangles) $a \sqsubseteq b$ means $a \supseteq b$. For $a \ll b$ we should have $\mathrm{Int}(a) \supseteq b$.

(3) Note that the 3-dim. example given in (2) does have dim = 3. Thus a very similar lattice where we use rectangles and compute as follows:



is not the same lattice because (I think) it obvious has dim = 4.

Anyway the point of these examples is to interest you in the function spaces which as yet you have not found algebraic enough.

I think we can make an analogy with the ring $C(X)$ of continuous real-valued functions. Even if we are only interested in rings, we take all continuous $f: X \to \mathbb{R}$ because we are using the pointwise operations on functions. It should be the same with the lattices. In any case, neither of the categories $\mathcal{CL}$ or $\mathcal{SL}$ are Cartesian closed. Of course $\mathcal{PO}$, the category of posets and monotone maps is cartesian closed, but it is too large. By using continuous lattices with all continuous maps (with or without $\top$) we have just the right Cartesian closed category without too many spaces. One especially important point is that if S and T have a countably based T_0-topology, then so does the function space $[S \to T]$ as a continuous lattice. I return to this below, but I think you agree that we need such control over the topology.

By the way, if we have a countable basis $\{x \mid a_n \ll x$ for the T_0-topology, then we have a countable basis for the Hausdorff topology because

$$\{x \mid b \not\sqsubseteq x\} = \bigcup \{\{x \mid a_n \not\sqsubseteq x\} \mid a_n \sqsubseteq b\}$$

because with a basis every element b satisfies:

$$b = \bigsqcup \{a_n \mid a_n \ll b\} = \bigsqcup \{a_n \mid a_n \sqsubseteq b\}.$$

I bet that a countable basis for the Hausdorff topology also implies a countable basis for the T_0-topology because I guess that the formula

$$b = \bigsqcup \{\sqcap \mathcal{U} \mid b \in \mathcal{U}\}$$

holds when $\mathcal{U}$ are basic Hausdorff sets and $\sqcap \mathcal{U} \ll b$. (This needs to be checked). If so, the notion of a separable continuous lattice is clear to us both (from both topologies) and it follows from my previous work that if S and T are separable, then so is $[S \to T]$. Clearly that is desirable and is a point in favour of my category.

Speaking of categories, I was very happy to see from ATLAS, that when S and T are both Lawson and g and d are adjoint, then both g and d are continuous from below (i.e. in my category). This is good because then dg and gd are both in my category, and, as far as I can see, neither of these maps are in your two categories CL or SL. The map $dg: S \to S$ is a retraction (indeed, projection) in my sense (as you show in detail), but it is not a semilattice morphism. The "morphic" equation that every projection satisfies is (when $j: S \to S$):

$$j(a \sqcap b) = j(j(a) \sqcap j(b)).$$

The point is that the range $j(S)$ of j is a continuous lattice in itself (a retract of S as I remarked), but it is not usually a subsemilattice of S (not closed under $\sqcap$).

In case $g: S \to T$ and $d: T \to S$ are adjoint and S and T are Lawson, then the range of g is easily seen to be isomorphic to the range of dg. This gives a quick proof that the quotient of a compact Lawson semilattice is again such, because I already knew that the retract of a continuous lattice is again such. You give this result as 5.3 on p. 84

The fact that morphisms $L \to I$ separate points is also rather easily proved from basic principles — by essentially your proof but without the later theory. Thus:

Define $a \ll b$ lattice theoretically by saying for every upward directed set D, whenever $b \sqsubseteq \bigsqcup D$ then $a \sqsubseteq x$ for some $x \in D$. The lattices we want are those where for all b:

$$b = \bigsqcup \{a \mid a \ll b\}.$$

If this is so then also

$$b = \bigsqcup \{\bigsqcup\{c \mid c \ll a\} \mid a \ll b\}$$
$$= \bigsqcup \{c \mid \exists a.\ c \ll a \ll b\}$$

Thus if $d \ll b$, then $d \sqsubseteq c \ll a \ll b$ and so $d \ll a \ll b$. This means that the ordering $\ll$ is dense over the whole lattice (you seem to make this a rather special point).

N.B. Actually, after trying to write it out I do not think your proof is correct! Your definition of $d : I \to I$ is

See p 86 of ms of ATLAS

$$d(t) = \bigsqcup\{ d_0(q) \mid q \leq t, q \in Q_0 \}$$

where $d_0 : Q_0 \to I$ is a certain function. The trouble is that as far as I can see $d(t)$ is not continuous (under sups in I). The function that is continuous is

$$d_1(t) = \bigsqcup\{ d_0(q) \mid q < t \}$$

with the strict inequality. This need not be an extension of d_0, but I think everything is OK if we are careful.

Begin slowly: take a pair of points where $a \not\sqsubseteq b$. Now there will be a $c \ll a$ with $c \not\sqsubseteq b$. We have $\bot \ll c \ll a$. By dyadic rationals take

$$d_0(0) = \bot, \quad d_0(1/2) = c, \quad d_0(1) = a$$

and by density interpolate elements so that $q < r$ implies:

$$d_0(q) \ll d_0(r).$$

Now define d_1 as above. We see $d_1(1) \sqsubseteq a$, but $d_1(1) \not\sqsubseteq b$ because $d_0(1/2) \sqsubseteq d_1(1)$. Also if $t < u$ in I, then $t < q < r < u$ for some $q, r \in Q_0$. Thus

$$d_1(t) \sqsubseteq d_1(q) \sqsubseteq d_0(q) \ll d_0(r) \sqsubseteq d_1(u).$$

Hence $d_1(t) \ll d_1(u)$. We can now apply the early 1.20 to get the adjoint g to d_1 where $g : I \to I$ and

$$g(x) = \bigsqcup\{ t \mid d_1(t) \sqsubseteq x \}.$$

We find $g(a) = 1$ but $g(b) \leq 1/2$. So g separates a and b. This seems correct (yes?) and more elementary than your method! (We can also see why we need $[0,1]$ and not just $\{0,1\}$.)

Turning now for a moment to the representation of $I \in \mathcal{CL}$ by the PI reflector, there is another side to the story that may interest you. The method in its final form is due to my student Mike Smyth which improves my ideas about giving bases for these lattices. Consider a structure $\langle H, \bot, \vee, < \rangle$ where we assume:

$$\bot \vee a = a \qquad a \vee b = b \vee a$$
$$a \vee a = a \qquad a \vee (b \vee c) = (a \vee b) \vee c$$

and

$a < b < c \Rightarrow a < c$
$a \leq b < c \Rightarrow a < c$
$a < b \leq c \Rightarrow a < c$

$\perp < c$
$a < c \;\&\; b < c \Rightarrow a \vee b < c$
$a < c \Rightarrow \exists b\,[a < b < c]$

(Here: $a \leq b \Leftrightarrow a \vee b = b$)

These are very obvious but weak axioms: $\langle H, \perp, \vee \rangle$ is a lower semilattice (later: hemilattice as I tell you below) and $<$ is a rather weak kind of strict ordering. Thus we do not assume $a \not< a$ and so $a < b \Leftrightarrow a \leq b$ is a possible interpretation for $<$ on $\langle H, \perp, \vee \rangle$. Good.

Now make dedekind cuts. Or better: first let

$$\mathcal{P}(H) = \{x \subseteq H \mid \perp \in x \;\&\; \forall a, b \in x.\; a \vee b \in x \;\&\; \forall a \in x\, \forall b < a.\; b \in x\}$$

Clearly $\mathcal{P}(H)$ is an algebraic lattice. Define the projection $j : \mathcal{P}(H) \to \mathcal{P}(H)$ by:

$$j(x) = \{a \mid \exists b \in x.\; a < b\}$$

This is continuous, its values lie in $\mathcal{P}(H)$ because $\perp < \perp$ and $a_1 < b_1$ and $a_2 < b_2$ implies $a_1 \vee a_2 < b_1 \vee b_2$, and $j(x) \subseteq x$. Also $j(x) = j(j(x))$ by the density of $<$. So j is a projection and its range is a continuous lattice: those "cuts" in $\mathcal{P}(H)$ where $a \in x \Rightarrow \exists b \in x.\; a < b$ (i.e. $x \subseteq j(x)$).

Now any $\mathbb{L}$ can be obtained in this way: take $\mathbb{L} = H$, $\perp = \perp$, $\vee = \sqcup$, $< = \ll$ and we have the above axioms. The range of j is in a one-one correspondence with $\mathbb{L}$ since

$$x \longleftrightarrow \{a \mid a \ll x\}$$

This is all very similar to what you do. But my point is that H can be much more elementary that $\mathbb{L}$. For example H can be countable while the completion by cuts is uncountable. (This analogy to cuts in the rationals was one of the first ideas I had about these lattices.)

What Smyth saw was just the right axioms on H so that we have enough to do cuts, but where it makes good sense to say: Well, suppose H is recursive. I mean, H is so countable that $H = \{0, 1, 2, 3, \dots\}$ and

The operations and relations $\vee$, $<$ (also $=$, $\leq$) are recursive. When this happens, we can say that

$$\mathbb{L} = j(\Phi(H))$$

is an <u>effectively given</u> continuous lattice. This is important as a concept for many, many reasons.

Note that in $j(\Phi(H))$ we have

$$x \sqcup y = \{c \mid \exists a \in x\, \exists b \in y.\ c < a \vee b\}$$
$$x \sqcap y = \{b \mid \exists a \in x \cap y,\ b < a\}$$

Assuming that H is recursive, the lattice structure on $\mathbb{L}$ is "computable" in the precise sense that there is a uniform way (by the above equations) to go from an enumeration of x and an enumeration of y to an enumeration of $x \sqcup y$ (or of $x \sqcap y$). And these are not the only computable functions on such a lattice.

What is more, if $\mathbb{L}_1$ and $\mathbb{L}_2$ are effectively given then so are $\mathbb{L}_1 \times \mathbb{L}_2$ and the function space $\mathbb{L}_1 \to \mathbb{L}_2$. ⊛ Thus, since the obvious maps are computable, the category of effectively given continuous lattice and computable (continuous!) maps is cartesian closed.

It seems to me that this ought to interest you: there is plenty of room in this category for computable semigroup morphisms. Indeed in an adjoint pair d, g if d is computable then so is is g. (I think). So this should give lots and lots of computable g.

To return to hemilattices: there is no need for H to be closed under $\vee$; that is, $\langle H, \perp, \vee \rangle$ could be a <u>partial</u> algebra. Thus in the axioms: $\perp \vee a$ and $a \vee a$ should always exist, but $a \vee b$ exists (in H) iff $b \vee a$ exists. Similarly $a \vee (b \vee c)$ exists iff $(a \vee b) \vee c$ exists and if, say, $a \vee (b \vee c)$ exists, then so does $b \vee c$. When $a < c$ & $b < c$, then $a \vee b$ has to exist. It is all very natural. The completion by cuts of H is now a lattice without necessarily a unit.

⊛ This last about $\mathbb{L}_1 \to \mathbb{L}_2$ is not so easy to prove. This was Smyth's real improvement over earlier results.

By the way, when I say "completion" I also allow "quotient". The mapping:

$$a \rightsquigarrow \{b \mid b < a\} \in j(\mathcal{P}(H))$$

is not always one-one. This too was Smyth's idea: I had tried to choose H as a sub-poset of I, but this seems to be <u>wrong</u> if you care about the effective structure on H.

A few points from your last letter. Suppose $L \underset{g}{\overset{d}{\rightleftarrows}} L'$ are an adjoint pair. Let $L \to L$ and $L' \to L'$ be the two function spaces of all continuous functions; for $f: L \to L$ and $f': L' \to L'$ we have

$$f \sqsubseteq g \circ f' \circ d \quad \text{iff} \quad d \circ f \circ g \sqsubseteq f' \qquad (*)$$

Thus we have adjointness between function spaces. What more do you want?? The function spaces <u>are</u> nice lattices; (Pointwise $\sqsubseteq$ and $\sqcap$ remember!); the obvious adjointness things go over — <u>because</u> we use functions continuous from below systematically (as in my theory); there are plenty of semigroup morphisms of the usual kind. So fine! There is no question of <u>explaining</u> function spaces by duality; rather <u>apply</u> duality to function spaces as above.

As you can see: it is late and my handwriting is becoming poorer and poorer. So enough for this time.

With all best regards,

Dana Scott.

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(*) The combination $g \circ f' \circ d$ shows again why we have to use <u>arbitrary</u> continuous maps and not just maps in $\mathcal{CL}$ or $\mathcal{SL}$.