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TOPIC Generalized Continuous Lattices

REFERENCE

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[4] Hofmann + Stralka, ATLAS

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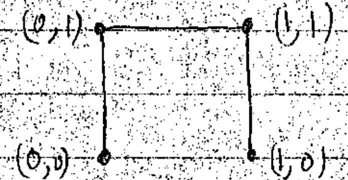
1 Generalized Continuous Lattices

We give here a generalization of continuous lattices which we stumbled upon in another context (to which we turn later on). Besides having application to later situations, these objects appear worthy of study in their own right. Their theory bears rather striking resemblances to that of continuous lattices and gives a new perspective to earlier work.

1.1 DEFINITION. Let L be a complete lattice. If $F \subset L$ is finite and $x \in L$, we write $F \ll x$ if for every up-directed set D, $\sup D \geq x$ implies $y \leq d$ for some $d \in D$, $y \in F$. In this case we say F <u>guards</u> x (<u>from below</u>). The idea here is that x cannot be penetrated without overrunning some member of F. $\square$

The following is our original example.

1.2 EXAMPLE. Consider the following subset of the square ordered by coordinatewise ordering, $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.



Thus L is an upside down U; L is a complete lattice w.r.t. the induced order although the meet operation is somewhat peculiar. Now let $F = \{(1, \frac{1}{2}), (\frac{1}{2}, 1)\}$. Then $F \ll 1$ $(= (1,1))$ and as a matter of fact such sets (but not singletons) can be picked arbitrarily close to 1.

Note that sets of the form $\{(\frac{1}{2}-\varepsilon, 1), (1, 1-\varepsilon)\}$ guard the point $(\frac{1}{2}, 1)$ and that the second points are necessary to prevent rear attacks. ▯

We turn now to the definition of a generalized continuous lattice. The idea is that each point can be guarded from finitely many locations "arbitrarily near" to $\uparrow x$.

1.3 DEFINITION. Let L be a complete lattice. Then L is a <u>generalized continuous lattice</u> (henceforth denoted gCL) if for all $x, y \in L$ such that $x \nleq y$, there exists a finite set $F \ni F \ll x$ and $\downarrow y \cap F = \phi$. ▯

This definition is a smoother version of our first one which we include in the following proposition (whose proof is quite straightforward and hence omitted).

1.4 PROPOSITION. Let L be a complete lattice. For each $x \in L$, let $\mathcal{F}_x = \{F \subset L : |F| < \infty \text{ and } F \ll x\}$. TAE

(1) L is a gCL;

(2) For each $x \in L$ and for each choice function $\alpha \in \prod_{F \in \mathcal{F}_x} F$,

$$\text{we have } x \leq \sup\{\alpha(F) : F \in \mathcal{F}_x\}. \quad ▯$$

One of the key properties of continuous lattices which makes everything work nicely is the interpolation property for $\ll$. We need also an interpolation property for gCL's, but it is more elusive in this setting. Hence we need to introduce and develop additional machinery.

Let X be a T_0 topological space. Then X has a partial ordering induced by defining $x \le y$ iff $y \in \mathrm{cl}\{x\}$. Conversely let $(X, \le)$ be a poset. Then this ordering induces a topology by defining all sets of the form $\uparrow x$ to be a subbase for the closed sets. The ordering induced by this topology is precisely the original ordering. This topology is not new, but seems to have no standard name. In [1] it is called the INF-topology while in [2] it is called the closure of points (COP) topology. It seems to be an obvious adaptation to partially ordered sets of the cofinite topology on sets, so we adopt the name <u>cofinite topology</u>.

~~We could call a topology on a poset $(X, \le)$ compatible if for all $x \in X$, $\mathrm{cl}\{x\} = \uparrow x$, i.e. the order induced by the topology is precisely the given order. Then the cofinite topology is the coarsest compatible topology while the dual Scott topology is the finest compatible topology.~~

Let X be a topological space, $A \subset B \subset X$. We say that A is <u>precompact</u> in B if every open cover of B contains a finite subcover of A. The following is a mild generalization of Alexander's Lemma.

1.5 PROPOSITION. Let $\mathcal{S}$ be a subbasis for the topology on X. If A is precompact in B for the subbasis $\mathcal{S}$, then A is precompact in B.

Proof. Suppose A is not precompact in B. Then there exists an ultrafilter $\mathcal{F}$ with $A \in \mathcal{F}$ such that $\mathcal{F}$ does not cluster (equivalently, converge) to any point in B. Hence for each point $x \in B$, there exists a basic open set U_x such that $x \in U_x \notin \mathcal{F}$. Since U_x is basic, there exist subbasic open sets $S_1, \ldots, S_n \in \mathcal{S}$ such that $U_x = S_1 \cap \ldots \cap S_n$. If each $S_i \in \mathcal{F}$, then $U_x \in \mathcal{F}$. Hence $\exists\, S_x \in \mathcal{S}$ such that $x \in S_x$, $S_x \notin \mathcal{F}$. Since B is precompact in A with respect to $\mathcal{S}$, there exist $S_1, \ldots, S_k \in \mathcal{S} \ni A \subset \bigcup_{i=1}^{k} S_i$, but $S_i \notin \mathcal{F}$, $\forall i$. Thus $X \setminus S_i \in \mathcal{F}$ $\forall i$, and hence $\emptyset = A \cap \bigcap_{i=1}^{k} (X \setminus S_i) \in \mathcal{F}$, a contradiction. $\square$

We now present a mild generalization of Definition 1.1.

1.6 DEFINITION. Let L be a complete lattice, let $F, G \subset L$. ~~[illegible]~~ Then $F \ll G$ if $\sup D \geq g$ for some directed set D and $g \in G$, then $y \leq d$ for some $y \in F$, $d \in D$. In this case F is said to guard G.

1.7 PROPOSITION. Let L be a complete lattice, $F, G \subset L$. Then $F \ll G$ iff $L \setminus \uparrow F$ is precompact in $L \setminus \uparrow G$ for the cofinite topology on L.

Proof. Suppose $L \setminus \uparrow F$ is precompact in $L \setminus \uparrow G$. Let D be a directed set such that $\sup D \geq x$ for some $g \in G$. Then $L \setminus \uparrow G \subset \bigcup \{L \setminus \uparrow d : d \in D\}$, since $\uparrow G \supset \uparrow g \supset \bigcap \{\uparrow d : d \in D\}$. Since each $L \setminus \uparrow d$ is open in the cofinite topology, there exist $d_1, \ldots, d_n \in D$ such that $L \setminus \uparrow F \subset \bigcup_{i=1}^{n} L \setminus \uparrow d_i$. Let $d \in D$ such that $d_1, \ldots, d_n \leq d$. Then $L \setminus \uparrow F \subset L \setminus \uparrow d$, i.e. $\uparrow d \subset \uparrow F$ and hence $d \geq f$ for some $f \in F$.

Conversely suppose $F \ll G$. Let $\{L \setminus \uparrow x\}_{x \in A}$ be a collection of subbasic open sets which covers $L \setminus \uparrow G$. Let D be the up-directed set of all sups of finite subsets of A. If $B \subset A$, $|B| < \infty$, then $L \setminus \uparrow(\sup B) \supset \bigcup_{b \in B} L \setminus \uparrow b$. Hence $\{L \setminus \uparrow d : d \in D\}$ is a cover of $L \setminus \uparrow G$. Thus $\sup D \in \uparrow G$. Since $F \ll G$, there exists $y \in F$ and a finite set $B \subset A$ such that $y \le \sup B$. Then

$$L \setminus \uparrow F \subset L \setminus \uparrow y \subset L \setminus \uparrow(\sup B) = \bigcup_{b \in B} L \setminus \uparrow b.$$

Hence finitely many of the collection $\{L \setminus \uparrow x : x \in A\}$ cover $L \setminus \uparrow F$. Thus by Proposition 1.5 $L \setminus \uparrow F$ is precompact in $L \setminus \uparrow G$. $\square$

1.8 THEOREM. Let L be a ~~complete~~ [illegible] generalized continuous lattice.

A. If $F \ll A$ and $G \ll A$, then $F \vee G \ll A$ (where $F \vee G = \{x \vee y : x \in F, y \in G\}$). For all F, G, $\uparrow(F \vee G) = \uparrow F \cap \uparrow G$.

~~(where $F \vee G = \{x \vee y : x \in F, y \in G\}$) and $\uparrow(F \vee G) \subset \uparrow F \cap \uparrow G$. If F and G are finite, then $\uparrow F \cap \uparrow G = \uparrow (F$~~

B. If A ~~[illegible]~~ is closed in the cofinite topology and $B \ll A$, then there exists a finite set F such that $A \subset \uparrow F \subset \uparrow B$ and furthermore $B \ll F \ll A$. In particular if $x \in L$, and G is a finite set such that $G \ll x$, then there exists a finite set F such that $G \ll F \ll x$ (the interpolation property).

Proof. A. Let $F, G \ll A$ and let D be a directed set such that $a \le \sup D$ for some $a \in A$. Then $\exists\, x \in F$ and $y \in G$ and $d_1, d_2 \in D \ni x \le d_1$ and $y \le d_2$. Pick $d \in D \ni d_1, d_2$, are less than or equal to d. Then $x \vee y \le d$. The last part of A. is straightforward.

B. Suppose that A is closed in the cofinite topology, and that $B \ll A$. Let

$\mathcal{F} = \{\uparrow F : F$ is finite and $\exists$ a finite set $G \ni F \ll G \ll A\}$.

We first show $A = \bigcap \mathcal{F}$. Let $y \in L \setminus A$. Since A is closed, A is the intersection of sets of the form $\uparrow G$ where G is finite. Hence there exists some set G which is finite such that $y \notin G$ and $A \subset \uparrow G$. Since L is a gCL, for each $g \in G$, $\exists$ a finite set F_g such that $F_g \ll g$ and $y \notin \uparrow F_g$. Let $F = \bigcup\{F_g : g \in G\}$. Then $F \ll G$ and $y \notin \uparrow F$. Repeat this process with F to obtain an $F_1 \ll F$ $\ni y \notin \uparrow F_1$. Then $F_1 \ll F \ll A$, and hence $F_1 \in \mathcal{F}$. Thus $y \notin \bigcap \mathcal{F}$. Since y was arbitrary, we have $A = \bigcap \mathcal{F}$.

Since each member of $\mathcal{F}$ is closed in the cofinite topology and $A = \bigcap \mathcal{F}$, $\{L \setminus \uparrow F : \uparrow F \in \mathcal{F}\}$ is an open cover of $L \setminus A$. By Proposition 1.7 $L \setminus \uparrow B$ is precompact in $L \setminus A$. Hence $\exists\, F_1, \ldots, F_n$ such that each F_i is finite, $\uparrow F_i \in \mathcal{F}$, and $L \setminus \uparrow B \subset \bigcup_{i=1}^{n} L \setminus \uparrow F_i$.

For each i, $\exists\, G_i$ finite such that $F_i \ll G_i \ll A$. Let $F = F_1 \vee \ldots \vee F_n$ and $G = G_1 \vee \ldots \vee G_n$. By part A. we have $G \ll A$ and $F \ll G$. Since $\uparrow F = \bigcap_{i=1}^{n} \uparrow F_i$, $L \setminus \uparrow B \subset L \setminus \uparrow F$. Hence $\uparrow F \subset \uparrow B$.

For the final part let $A = \uparrow x$. $\square$

1.9 COROLLARY. Let L be a complete lattice. TAE

(1) L is a gCL;

(2) The lattice $O_{cf}(L)$ of sets open in the cofinite topology is a continuous lattice (with respect to the operation of intersection).

Proof (1) ⇒ (2). Let U be open in the cofinite topology, and let $x \in U$. Then $L \setminus U$ is closed and $x \notin L \setminus U$. Hence as in the proof of Theorem 1.8, $\exists$ a finite set F such that $L \setminus U \subset \uparrow F$ and $F \ll L \setminus U$ (equivalently $\uparrow F \ll L \setminus U$) and $x \notin \uparrow F$. By Proposition 1.7 we have $x \in V = L \setminus \uparrow F \subset U$ and that V is precompact in U. Since x was arbitrary $U = \bigcup \{V : V \text{ is precompact in } U\}$. This is precisely the condition needed for $O_{cf}(L)$ to be a continuous lattice.

(2) ⇒ (1). Let $x, y \in L$, $x \nleq y$. Then $y \in L \setminus \uparrow x = U$, which is open in the cofinite topology. By hypothesis there exists an open set V such that $y \in V$ and V is precompact in U. Since $O_{cf}(L)$ is a continuous lattice and hence has the interpolation property, $\exists$ an open set W such that V is precompact in W and W is precompact in U. Since $K = L \setminus W$ is closed, as in the proof of the preceding theorem K is the intersection of upper sets of finite sets, and this is a descending family. Since V is precompact in W, there exists a finite set F such that $K \subset \uparrow F$ and $V \subset L \setminus \uparrow F$. Thus $y \notin \uparrow F$ and by Proposition 1.7 we have $K \ll x$ and hence $F \ll x$ since $K \subset \uparrow F$. Thus L is a gCL. □

2. Topologies on Generalized Continuous Lattices

2.1 DEFINITION. Let $(X, \leq)$ be a partially ordered set. A topology $\mathcal{O}$ on X is said to be <u>order compatible</u> if (i) for all $x \in X$, $\mathrm{cl}(\{x\}) = \uparrow x$, and (ii) if D is a descending subset of X and $z = \inf D$, then considering D as a net, D converges to z in $\mathcal{O}$. $\square$

If X is a T_0 space, then a partial order may be defined on X by $x \leq y$ if $y \in \mathrm{cl}(\{x\})$. If a topology on a partially ordered set is order compatible, then this induced order is precisely the original order.

2.2 PROPOSITION. On a partially ordered set $(X, \leq)$ the cofinite topology is the coarsest of the order compatible topologies and the dual Scott topology is the finest.

<u>Proof</u>. Since for every closed set A in the cofinite or dual Scott topology $\uparrow A = A$ and since $\uparrow x$ is closed for all x, we have $\mathrm{cl}(\{x\}) = \uparrow x$ for both of these topologies.

Let D be a descending subset of X. If D does not converge to $z = \inf D$ in the cofinite topology, then $\exists$ a subbasic open set $L \setminus \uparrow x$ such that $z \in L \setminus \uparrow x$ but some cofinal subset of D misses $L \setminus \uparrow x$. But since D is descending, we have that $\exists\, d \in D$ such that $d' \in \uparrow x$ for $d' \leq d$. Hence $x \leq \inf D$, i.e. $x \leq z$. This contradicts $z \in L \setminus \uparrow x$. Hence the cofinite topology is order compatible.

If D does not converge to z in the d-Scott topology, then there exists an open set U such that $z \in U$, but some cofinal subset of D misses U. Since $U = \downarrow U$, D misses U. Since U is open, $z = \inf D \notin U$, a contradiction. Thus the dual Scott topology is order compatible.

Now let $\mathcal{U}$ be any order compatible topology. Since for all $x \in X$, $\uparrow x = \mathrm{cl}_{\mathcal{U}}(\{x\})$, which is closed, the topology $\mathcal{U}$ is finer than the cofinite topology. Let C be a closed set in X for the $\mathcal{U}$-topology. Then $\uparrow C = C$ (from condition (i) for order compatibility). Let D be a down-directed set, $D \subseteq C$. Then D converges to $\inf D$ since $\mathcal{U}$ is order compatible and hence $\inf D \in C$ since C is closed. Thus C is closed in the dual Scott topology. $\square$

We return to these "one-sided" topologies shortly, but first we have need to introduce some "two-sided" ones.

2.3 DEFINITION. Let L be a complete lattice. We define the CL (or Lawson) topology on L to be the topology for which all Scott open and all cofinite open sets form a subbase. We define the Lim Inf topology (LI) by declaring a set A to be closed if every ultrafilter which has A as a member has the liminf of the ultrafilter in A (if $\mathcal{F}$ is an ultrafilter, $\liminf \mathcal{F} = \sup\{\inf F : F \in \mathcal{F}\}$). Equivalently A is closed if for every universal net in A, the liminf of the net is again in A. $\square$

2.4 PROPOSITION. Let L be a complete lattice. The CL and LI topologies are compact and T_1. The identity mapping is continuous from (L, LI) to (L, CL).

Proof. We first show the latter statement. Let K be a closed set in the cofinite topology. Let $\mathcal{F}$ be an ultrafilter such that $K \in \mathcal{F}$. Let M be a finite set such that $K \subset \uparrow M$. It suffices to show $\liminf \mathcal{F} \in \uparrow M$ since by definition of the cofinite topology K is the intersection of such sets. Since $\mathcal{F}$ is an ultrafilter and since $\uparrow M = \bigcup\{\uparrow x : x \in M\}$ and this latter collection is finite, $\exists\, x \in M$ such that $\uparrow x \in \mathcal{F}$. Thus $x \leq \liminf \mathcal{F}$ and hence $\liminf \mathcal{F} \in \uparrow M$.

Now let A be a Scott closed set and let $\mathcal{F}$ be an ultrafilter such that $A \in \mathcal{F}$. For any $F \in \mathcal{F}$, $F \cap A \neq \emptyset$, and hence $\inf F \leq a$ for some $a \in A$. Since $A = \downarrow A$, $\inf F \in A$. Since F contains sups of up-directed sets, $\liminf \mathcal{F} \in A$. Thus both cofinite closed sets and Scott closed sets are closed in the LI topology. Hence the identity function is continuous.

Let L be equipped with the LI topology. Then in this topology ultrafilters still converge to their limit infima (a general characteristic of defining a topology in terms of filters or nets), although additional limit points may also exist in the topology. In particular since every ultrafilter has a point of convergence, (L, LI) is compact. By continuity (L, CL) is compact.

Now (L, CL) is T_1 since $\{x\} = \uparrow x \cap \downarrow x$ and $\uparrow x$ is closed in the cofinite topology and $\downarrow x$ is closed in the Scott topology. Hence by continuity of the identity mapping (CL, LI) is T_1. $\square$

We come at this point to a major theorem.

2.5 THEOREM. Let L be a complete lattice. TAE

(1) L is a gCL;

(2) (L, CL) is Hausdorff;

(3) (L, LI) is Hausdorff.

Furthermore if any of these equivalent conditions are satisfied,

~~[illegible]~~

then the LI and CL topologies agree and the partial order $\leq$ has closed graph for this topology. Furthermore, this topology has as a subbase of open sets all sets of the form $\{s : F \ll s\}$ where F is some finite set and $L \setminus \uparrow x$ where $x \in L$.

Proof. (1) $\Rightarrow$ (2) Let L be a gCL. Let $x, y \in L$ and suppose that $x \not\leq y$. Then $\exists$ a finite set F such that $F \ll x$ and $y \notin \uparrow F$. Let $U = \{s : F \ll s\}$ and let $V = L \setminus \uparrow F$. Then $U \cap V = \emptyset$ and V is open in the cofinite, and hence CL, topology. To finish the proof we show U is Scott open. Let D be an up directed set such that $p = \sup D \in U$. By the interpolation property (1.8 B), $\exists$ a finite set G such that $F \ll G \ll p$. Thus there exists $d \in D$ such that $b \leq d$ for some $b \in G$. But $F \ll G$ implies $F \ll b$ and thus $F \ll d$. Hence $\sup D \in U$ implies $d \in U$ for some $d \in D$. Thus U is Scott open.

Note that by the preceding paragraph sets of the form $\{s : F \ll s\}$ and $L \setminus \uparrow x$, $x \in L$ generate a Hausdorff topology if L is a gCL, and that these sets are open in the CL topology. Since by Proposition 2.4 the CL topology is compact, these sets must generate precisely the CL topology. Hence the last statement of the theorem holds.

(2) ⇒ (3) Immediate since by 2.4 the identity function from (L, LI) to (L, CL) is continuous.

(3) ⇒ (1) Note first that if $\mathcal{F}$ is an ultrafilter on L, then $\mathcal{F}$ converges to $\liminf \mathcal{F}$ and to that point alone in the LI topology (since convergence by definition implies convergence in the topology and since by Hausdorffness there is at most one point of convergence).

We show that the relation $\leq$ is closed in $L \times L$ (with each factor equipped with the LI topology). Let $\mathcal{G}$ be an ultrafilter contained in $\{(x, y) : x \leq y\}$, i.e., the set $\leq$ is a member of $\mathcal{G}$. Let $\mathcal{G}_1 = \{\pi_1(G) : G \in \mathcal{G}\}$ and $\mathcal{G}_2 = \{\pi_2(G) : G \in \mathcal{G}\}$. For each $G \in \mathcal{G}$ such that $G \subset \{(x, y) : x \leq y\}$, we have, for the first and second projections, $\pi_1(G)$ and $\pi_2(G)$, $\inf(\pi_1(G)) \leq \inf(\pi_2(G))$. Thus $\liminf\{\pi_1(G) : G \in \mathcal{G}\} \leq \liminf\{\pi_2(G) : G \in \mathcal{G}\}$. Since $\mathcal{G}$ is an ultrafilter, $\pi_1(\mathcal{G})$ and $\pi_2(\mathcal{G})$ are ultrafilters. Hence if $a = \liminf\{\pi_1(G) : G \in \mathcal{G}\}$ and $b = \liminf\{\pi_2(G) : G \in \mathcal{G}\}$, then $\pi_1(\mathcal{G})$ and $\pi_2(\mathcal{G})$ converge to a and b resp., and hence $\mathcal{G}$ converges to (a, b). Hence the relation $\leq$ is closed.

We have that L is a compact Hausdorff space with a closed partial order with respect to the LI topology. Hence we may invoke the known properties of such structures. Let $x, y \in L$ with $x \not\leq y$. We wish to find a finite set F such that $F \ll x$ and $y \notin \uparrow F$. It suffices to find a finite F such that $y \notin \uparrow F$ and there exists an open set U such that $U = \uparrow U$ and $x \in U \subset \uparrow F$ (for if D is an up-directed set with $\sup D \geq x$, then since D converges to its supremum, $d \in U \subset \uparrow F$ for some $d \in D$).

Thus suppose for every open set $U = \uparrow U$ with $x \in U$ and for every finite F such that $y \notin \uparrow F$, $U \not\subset \uparrow F$, i.e., $U \setminus \uparrow F \neq \emptyset$. These sets then form a filter base; extend this base to an ultrafilter $\mathcal{L}$. Since $\mathcal{L}$ contains all open neighborhoods $U = \uparrow U$ of x, the limit of $\mathcal{L}$ is a member of $\uparrow x$. Since $\{\inf L : L \in \mathcal{L}\}$ converges to the liminf of $\mathcal{L}$ which must be the limit of $\mathcal{L}$, there exists $L \in \mathcal{L}$ such that $\inf L \not\leq y$. We thus have $\uparrow z \in \mathcal{L}$ where $z = \inf L$ and $L \setminus \uparrow z \in \mathcal{L}$ from the original definition. However this is impossible, so the argument is complete.

We have shown in this argument that $\leq$ is closed. Since (L, LI) is compact, if (L, CL) is Hausdorff, then the two topologies agree by 2.4. $\square$

2.6 COROLLARY. Let L be a meet continuous complete lattice. Then if L is a gCL, it is a continuous lattice.

Proof. By [ ] a meet continuous complete lattice for which the CL topology is Hausdorff is a continuous lattice. Hence the corollary follows from Theorem 2.5.

We return now to a more detailed consideration of the cofinite topology.

2.7 PROPOSITION. Let L be a complete lattice.
A. If $\mathcal{F}$ is an ultrafilter on L, the the set of cluster (= convergence) points is $\uparrow(\liminf \mathcal{F})$ (in the cofinite topology).
B. A subset M of L is closed in the cofinite topology iff $\uparrow M = M$ and for every ultrafilter $\mathcal{F}$ with $M \in \mathcal{F}$, $\liminf \mathcal{F} \in M$.
C. A subset M of L is closed in the cofinite topology iff $\uparrow M = M$ and M is closed in the CL-topology.

__Proof.__ A. The cluster points of an ultrafilter consists of all points in the intersection of the closure of all sets in the ultrafilter. If A is the set of cluster points, we have $A = \bigcap\{\overline{F} : F \in \mathcal{F}\} \subset \bigcap\{\uparrow(\inf F) : F \in \mathcal{F}\} = \uparrow(\liminf \mathcal{F})$. Conversely let $y \geq \liminf \mathcal{F}$. Let $F \in \mathcal{F}$. If $y \notin \overline{F}$, then by definition of the cofinite topology, $\exists$ a finite set K such that $\overline{F} \subset \uparrow K$ and $y \notin \uparrow K$. Since $F \subset \overline{F} \subset \uparrow K$, we have $\uparrow K \in \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, $\uparrow x \in \mathcal{F}$ for some $x \in K$. Hence $x \leq \liminf \mathcal{F}$, a contradiction since $x \not\leq y$. Thus $y \in \overline{F}$ for all $F \in \mathcal{F}$. Hence y is a cluster point.

B. Suppose M is closed in the cofinite topology. It follows immediately that $M = \uparrow M$. Since M is closed if $\mathcal{F}$ is any ultrafilter such that $M \in \mathcal{F}$, then it follows that all cluster points of $\mathcal{F}$ are contained in M. Hence by part A, we have $\liminf \mathcal{F} \in M$.

Conversely suppose $\uparrow M = M$ and $M \in \mathcal{F}$ implies $\liminf \mathcal{F}$ for every ultrafilter $\mathcal{F}$. Let $y \in \overline{M}$

Then $\exists$ an ultrafilter $\mathcal{F}$ with $M \in \mathcal{F}$ converging to y. By part A, $y \geq \liminf \mathcal{F}$. Since $\liminf \mathcal{F} \in M$ and $M = \uparrow M$, $y \in M$. Thus M is closed.

C. If M is closed in the cofinite topology, then it is closed in the CL topology by definition. Conversely suppose $M = \uparrow M$ and M is closed in the CL topology. Then by Proposition 2.4 it is closed in the LI topology and hence by part B in the cofinite topology. $\square$

2.8 COROLLARY. Let L be a complete lattice, and let $\mathcal{O}$ be an order compatible topology on L. The set of cluster points for an ultrafilter $\mathcal{F}$ is contained in $\uparrow(\liminf \mathcal{F})$.

Proof. This corollary follows easily from 2.7A and 2.2. $\square$

2.9 PROPOSITION. Let L be a complete lattice. A subset U of L is open in the Scott topology iff $U = \uparrow U$ and U is open in the CL topology. Furthermore the following are equivalent:

(1) L is a gCL;

(2) For every ultrafilter $\mathcal{F}$ the set of cluster points of $\mathcal{F}$ for the Scott topology is $\downarrow(\liminf \mathcal{F})$.

Proof. By definition of the CL topology, if U is Scott open then U is CL open. Conversely suppose $U = \uparrow U$ is CL open. Let D be an up directed set (with $x = \sup D$) in $L \setminus U$. It is easily verified that as a net D converges to every point of $\uparrow x$ in the cofinite topology, to every point of $\downarrow x$ in the Scott topology and hence precisely to x in the CL topology. Since $L \setminus U$ is CL closed, $x \in L \setminus U$. Hence $L \setminus U$ is Scott closed, i.e.

(1) ⇒ (2). Let $\mathcal{F}$ be an ultrafilter, $y = \liminf \mathcal{F}$. If $x \not\leq y$, then $\exists$ a finite set K such that $K \ll x$ and $y \notin \uparrow K$. Let $W = \{w : K \ll w\}$. As in the proof of Theorem 2.5 W is Scott open. Hence x is not a cluster point of $\mathcal{F}$ for the Scott topology.

Let $F \in \mathcal{F}$ and let $z \leq y$ with U a Scott open set such that $z \in U$. Since U is Scott open, $y \in U$, and hence $\inf F' \in U$ for some $F' \in \mathcal{F}$. Since $F' \subset \uparrow(\inf F')$, $F' \subset U$. Also $F \cap F' \neq \phi$ implies $F \cap U \neq \phi$. Since U was arbitrary, $z \in \bar{F}$, closure taken in the Scott topology. Since F was arbitrary, z is a cluster point of $\mathcal{F}$ (note that this inclusion holds even if L is not a gCL).

(2) ⇒ (1). By 2.7A, hypothesis, and the definition of the CL topology, the only possible cluster point for an ultrafilter $\mathcal{F}$ is $\liminf \mathcal{F}$. Hence the CL topology is Hausdorff. The conclusion follows from Theorem 2.5. []

2.10 PROPOSITION. Let L be a complete lattice equipped with a compact topology for which the closed upper sets are precisely the cofinite closed sets and for which the partial order is closed. The L is a gCL.

Proof. It is well known that the closed upper sets of a compact space with a closed order form a continuous lattice. The proposition follows from Proposition 1.9 []