3. Morphisms Between Generelized Continuous Lattices

In this section we want to describe under which conditions a mapping between to complete lattices is continous for the various topologies. In this context it seems to be reasonable to restrict ourselves to monoton mappings. Let us start with a well known result [ ]:

3.1. Proposition: Let $\varphi: L \to L'$ be a monoton mapping. The φ is continuous for the Scott-topologies on L and L' resp. iff φ preserves suprema of upwards directed families. ∎

3.2. Proposition: Let $\varphi: L \to L'$ be a mapping preserving upwards directed suprema. Then φ is continuous for the cofinite topologies on L and L' resp. iff φ preserves liminf's of ultrafilters. Moreover any monoton mapping preserving liminf's of ultrafilters is continuous.

Proof. Let us first assume that φ is continuous. Pick an ultrafilter $\mathfrak{F}$ on L. Then $\varphi(\liminf \mathfrak{F})$ is an clusterpoint of the ultrafilter $\varphi(\mathfrak{F})$ by the continuity of φ. This implies $\liminf \varphi(\mathfrak{F}) \leq \varphi(\liminf \mathfrak{F})$. Conversely, $\varphi(\liminf \mathfrak{F}) = \varphi(\sup\{\inf F : F \in \mathfrak{F}\})$ and the supremum is upwardsdirected. Hence $\varphi(\liminf \mathfrak{F}) = \sup\{\varphi(\inf F) : F \in \mathfrak{F}\} \leq$ $\leq \sup\{\inf \varphi(F) : F \in \mathfrak{F}\} = \liminf \varphi(\mathfrak{F})$ and this inequality holds because φ is monotan.

Now let φ be any monoton mapping preserving liminf's of ultrafilters. Then φ is continuous: Let $A' \subseteq L'$ be closed in the cofinite topologie and $A = \varphi^{-1}(A')$. Then A is an upper set because φ is monoton and A is closed under liminf's of ultrafilters containing A because A' is. Hence A is closed in the cofinite topology by (2.7) ∎

3.3 Proposition: Let $\varphi: L \to L'$ be a monoton mapping between two complete lattices L and L'. If φ preserves upwards directed suprema and liminf's of ultrafilters, then φ is continuous for the CL topologies on L and L'. Moreover if L is a generelized continuous lattice, then the converse holds also true.

Proof. By (3.1), (3.2) and the definition of the CL-topology we have only to give a proof of the second statement. Let us assume that L is a gCL and that $\varphi: L \to L'$ is CL-continuous. If $\mathcal{U} \subseteq L'$ is Scott open, then $\varphi^{-1}(\mathcal{U})$ is an open upperset and hence open in the Scott topology by (2.9). Hence φ is Scott continuous and therefore upward directed sup preserving by (3.1). Next let $a' \in L'$ be compact and $A = \varphi^{-1}(\uparrow a')$.

Then A is a CL-closed upper set of L and therefore closed in the cofinite topology on L by (2.7). Now we can conclude that φ is continuous in the cofinite topology and hence preserves liminf's of ultrafilters by (3.2). ■

In the remainder of this section we will show that generalized continuous lattices a preserved under a weak kind of quotient, subobjects and products.

3.4 PROPOSITION. Let L be a gCL, L' a complete lattice and $\varphi : L \to L'$ be a surjective mapping which preserves upwards directed suprema and which is continuous for the cofinite topology (i.e. φ is monoton and continuous for the CL-topologies). Then L' is a gCL.

Proof. Endow L with the liminf topology. Then (L, LI) is compact and T_2 by (2.5). Moreover, $\ker\varphi = \{(a,b) \in L \times L \; ; \; \varphi(a) = \varphi(b)\}$ is closed in $L \times L$: Let $\mathfrak{F}$ be an ultrafilter on $L \times L$ containing $\ker\varphi$. We have to show that $\lim \mathfrak{F} = (\lim \pi_1 \mathfrak{F}, \lim \pi_2 \mathfrak{F}) = (\liminf \pi_1 \mathfrak{F}, \liminf \pi_2 \mathfrak{F})$ is contained in $\ker\varphi$, where $\pi_i : L \times L \to L$ denotes the i^{th} projection, $i = 1,2$. First, note that $M \in \ker\varphi$ implies $\varphi \circ \pi_1 (M) = \varphi \circ \pi_2 (M)$ and that φ preserves liminf's of ultrafilters by the assumptions (3.4) and proposition (3.1). Hence $\varphi(\liminf \pi_1 \mathfrak{F}) = \liminf \varphi\pi_1 \mathfrak{F} = \liminf \varphi\pi_2 \mathfrak{F} = \varphi(\liminf \pi_2 \mathfrak{F})$. Now we can conclude that the quotient topology of φ on L' is T_2. Moreover, by (3.3), φ is continuous for the CL-topologies on L and L' resp. As the CL-topology on L coincides with the LI-topology by (2.5), the quotient topology on L' is coarser than the CL-topology on L'. Hence the CL-topology on L' is T_2. This finishes the proof by (2.5). ■

3.5 DEFINITION. A CL-morphism $\varphi : L \to L'$ between to complete lattices is a morphism preserving arbitrary infima and upwards directed suprema. ■

Clearly, every CL-morphism preserves liminf's of ultrafilters. This yields

3.6. PROPOSITION. Every CL-morphism is continuous for the cofinite topologies, the SCOTT-topologies, the CL-topologies and the LI-topologies on L and L' resp. ■

3.7 PROPOSITION: Let $\varphi: L \to L'$ be a CL-morphism between complete lattices L and L':

(i) If φ is surjective and if L is a gCL, then L' is a gCL.

(ii) If φ is injective and if L' is a gCL, then L is a gCL.

Proof. (i) follows from (3.4) and (3.6) and (ii) follow from the observation that $\varphi(L)$ is a subsemilattice of L, which is closed in the LI-topology. ∎

Now let L be a complete lattice and $Id(L)$ its ideal lattice.
From [ATLAS] we know that L is a continuous lattice if and only if $I \mapsto \sup I : Id(L) \to L$ is a CL-morphism and that L is meet continuous if and only if $I \mapsto \sup I : Id(L) \to L$ preserves finite infima. We add one more condition:

3.8. THEOREM. Let L be a complete lattice. Then L is a gCL if and only if $I \mapsto \sup I : Id(L) \to L$ is continuous for the cofinite topology (or equivalently for the CL-topology).

Proof. As the mapping $I \mapsto \sup I$ always preserves upwards directed suprema, one direction is clear by (3.4). Conversely, let L be a gCL and $a \in L$ be a point. We have to prove that $\{I \in Id(L) : a \neq \sup I\}$ is open in the cofinite topology of $Id(L)$. Let $J \in \{I \in Id(L) : a \neq \sup I\}$. Then $\sup J \neq a$. Hence we can find a finite set $F \subseteq L \setminus {\downarrow}\sup J$ which guards a from below. Now let $\mathcal{O} = \{I \in Id(L) \mid {\downarrow}f \not\subseteq I \text{ for all } f \in F\}$. Then $J \in \mathcal{O}$ and $\mathcal{O}$ is open in the cofinite topology of $Id(L)$. Moreover $I \in \mathcal{O}$ implies $\sup I \neq a$, because otherwise we would have ${\uparrow}f \cap I \neq \emptyset$ for some $f \in F$, and this $f \in I$, i.e. ${\downarrow}f \subseteq I$. Hence $J \in \mathcal{O} \subseteq \{I \in Id(L) : a \neq \sup I\}$. ∎

3.9 COROLLARY: A complete lattice L is generalized continuous if and only if it is the quotient of a continuous lattice L' under a mapping φ, which is continuous for the cofinite topologies on L and L' resp. and which preserves upwards directed suprema. ∎

We conclude this sections with two results, which may perhaps illustrate that many properties holding for continuous lattices are also true for generalized continuous lattice

3.10 PROPOSITION (c.w. [ATLAS]) If L and L' are gCL and $\gamma : L \to L'$ is a mapping preserving arbitrary infima, then γ preserves updirected suprema iff $F \ll G$ in L' implies $\partial F \ll \partial G$ in L, where $\partial : L' \to L$ is the right adjoint of γ (i.e. $\partial(a) = \inf \gamma^{-1}(\uparrow a)$)

Proof. Assume that γ preserves updirected suprema and that $\sup D \geq \partial(g)$ for some $g \in G$ and an upwards directed familie $D \subseteq L$. Then $\gamma(\sup D) = \sup \gamma(D) \geq g$, hence $\gamma(d) \geq f$ for some $d \in D$ and some $f \in F$. This implies $d \geq \partial(f)$ for some $d \in D$ and some $f \in F$, i.e. $\partial F \ll \partial G$.

Conversely, assume that $D \subseteq L$ is upwards directed. Let $F \subseteq L'$ be finite such that $F \quad \gamma(\sup D)$. Then we have $\partial F \ll \partial \gamma(\sup D) \leq \sup D$. By definition of $\ll$ this yields an $f \in F$ and an $d \in D$ such that $\partial f \leq d$, i.e. $f \leq \gamma d$. Hence we can conclude that $\sup \gamma D \in \bigcap \{\uparrow F \mid F \ll \gamma \sup D\} = \uparrow \gamma \sup D$, i.e. $\gamma \sup D \leq \sup \gamma D$. The other inequality holds for every monoton mapping. ■

Our second example is the Lemma on primes:

3.11 THE LEMMA ON PRIMES (c.w. [ ]) Let L be a complete lattice and L' be a gCL. Furthermore, let $\varphi : L' \to L$ be a mapping preserving upwards directed suprema and $p \in L$ be an element of L satisfying the primality condition

(P) For every finite subset $A \subseteq L'$, $\inf \varphi(A) \leq p$ implies the existence of an $a \in A$ such that $\varphi(a) \leq p$.

Then for every CL-compact subset $K \subseteq L'$, $\inf \varphi(K) \leq p$ implies the existence of an $k \in K$ such that $\varphi(k) \leq p$.

Proof. Let $K \subseteq L'$ be CL-compact such that $\inf \varphi(K) \leq p$ and assume that $K \cap \varphi^{-1}(\downarrow p)$ is empty. Then K is contained in the Scott open set $L' \setminus \varphi^{-1}(\downarrow p)$. Hence for every $k \in K$ we can find a finite subset $F_k \subseteq L' \setminus \varphi^{-1}(\downarrow p)$ with $F_k \ll k$ (If every finite subset $F \ll k$ would intersect $\varphi^{-1}(\downarrow p)$, then $\bigcap \{\uparrow F \cap \varphi^{-1}(\downarrow p) : F \ll k\}$ would be non empty, because L' is CL-compact. But $\bigcap \{\uparrow F \cap \varphi^{-1}(\downarrow p) : F \ll k\} = \varphi^{-1}(\downarrow p) \cap \bigcap \{\downarrow F : F \ll k\} = \varphi^{-1}(\downarrow p) \cap \uparrow k = \emptyset$.) Now we have $K \subseteq \bigcup_{k \in K} \{x : F_k \ll x\}$ and hence $K \subseteq \bigcup_{i=1}^{n} \{x : F_{k_i} \ll x\}$ for certain $k_1, \dots, k_n \in K$ by the compactness of K. (Recall from the proof of (2.5) that the set $\{x : F_k \ll x\}$ is open in the Scott-topology.) Now, clearly $p \geq \inf \varphi(K) \geq \inf \varphi(\bigcup_{i=1}^{n} F_{k_i})$ and $\bigcup_{i=1}^{n} F_{k_i}$ is finite. Hence the primality condition (P) yields an element $f \in \bigcup_{i=1}^{n} F_{k_i}$ with $\varphi(f) \leq p$, a contradiction to $\bigcup_{i=1}^{n} F_{k_i} \subseteq L' \setminus \varphi^{-1}(\downarrow p)$. Therefore we can find an element $k \in K \cap \varphi^{-1}(\downarrow p)$, i.e. $\varphi(k) \leq p$. ■

4. Complete Lattices in the Intervall Topologie

In this section we discuss the symmetric intervall topology and its relations to generalized continuous lattices.

4.1 Definition. Let L be a complete lattice. We define the IV (= intervall) topology on L to be the topology which has as a subbase of the closed sets the set of all closed intervalls $[a,b] = \{x \mid a \le x \le b\}$, $a,b \in L$. ∎

It is immediate that the IV-topology is the supremum of the cofinite topologies on L and L^{op} resp. This observation proves a large part of

4.2 Proposition. Let L be a complete lattice and $\mathfrak{F}$ an ultrafilter on L. Then the set of all clusterpoints of $\mathfrak{F}$ in the IV-topology is exactly the set $[\liminf \mathfrak{F}, \limsup \mathfrak{F}]$, and this set is not empty. Especially, L is a compact T_1 space in the IV-topology and the IV-topology is coarser than the LI-topologies and the CL-topologies on L and L^{op} resp.

Proof. Clearly, for every filter $\mathfrak{F}$ we have $\sup\{\inf M : M \in \mathfrak{F}\} \le \inf\{\sup M : M \in \mathfrak{F}\}$ and hence $[\liminf \mathfrak{F}, \limsup \mathfrak{F}] \neq \emptyset$. By (2.7) and the above remark, $[\liminf \mathfrak{F}, \limsup \mathfrak{F}]$ is exactly the set of all cluster points of an ultrafilter $\mathfrak{F}$. ∎

4.3 Proposition. Let L be a complete lattice. TAE

(i) The IV-topology on L is Hausdorff.

(ii) For every ultrafilter $\mathfrak{F}$ on L we have $\limsup \mathfrak{F} = \liminf \mathfrak{F}$.

(iii) L is a bi-generalized continuous lattice (i.e. L and L^{op} are gCL and the CL-topologies on L and L^{op} agree.)

In each of these cases, the IV-topology and the CL-topology agree.

Proof. (i) ⟺ (ii) holds by (4.2).

(i) ⇒ (iii) By (4.2), the IV-topology is coarser than both the CL-topology on L and the CL-topology on L^{op}. If the IV-topology is T_2, then all the three topologies agree. Hence (iii) follows by (2.5).

(iii) ⇒ (ii) is an easy consequence of (2.5). ∎

4.4. COROLLARY. Let L be a meet & join continuous lattice. Then L is bicontinuous iff the intervall topology on L is T_2 iff the CL-topology on L and the intervall topology on L agree. ∎

We will give an other description of bicontinuous lattices later on. For the moment let us return to lattices which are T_2 in their intervall topology. This type of lattices is preserved under a certain kind of quotients:

4.5. PROPOSITION. Let L and L' be to complete lattices and $\varphi : L' \twoheadrightarrow L$ be a surjective mapping preserving upwards directed suprema and downwards directed infima. Assume moreover that the IV-topology on L' is T_2. Then the intervall topology on L is also T_2.

Proof. Let $\mathfrak{F}$ be an ultrafilter on L and $\mathfrak{F}'$ be an ultrafilter on L' which is mapped onto $\mathfrak{F}$.
Then $\limsup \mathfrak{F} = \limsup \varphi(\mathfrak{F}') = \inf\{\sup \varphi(M) : M \in \mathfrak{F}'\} \leq \inf\{\varphi(\sup M) : M \in \mathfrak{F}'\} = \varphi(\inf\{\sup M : M \in \mathfrak{F}'\})$ $= \varphi(\limsup \mathfrak{F}') = \varphi(\liminf \mathfrak{F}') = \varphi(\sup\{\inf M : M \in \mathfrak{F}'\}) = \sup\{\varphi(\inf M) : M \in \mathfrak{F}'\} \leq \sup\{\inf \varphi(M) : M \in \mathfrak{F}'\} =$ $= \liminf \varphi(\mathfrak{F}') = \liminf \mathfrak{F}$, because $\limsup \mathfrak{F}' = \liminf \mathfrak{F}'$ in L' and because φ preserves upwards directed suprema and downwards directed infima. Hence $\liminf \mathfrak{F} = \limsup \mathfrak{F}$. ∎

COROLLARY. Let L' be a completely distributive lattice and L be a complete lattice. Further, let $\varphi : L' \twoheadrightarrow L$ be a surjective mapping preserving downward directed infima and upwards directed suprema. Then the intervall topology on L is T_2. ∎

4.7. PROPOSITION. Let L and L' be two complete lattice and let $\varphi : L \to L'$ be a monoton mapping. If the IV-topologies on L and L' resp. are T_2, then φ is continuous iff φ preserves upwards directed suprema and downwards directed infima.

Proof. Every supremum of an upwards directed family is the limit of the family, indexed by itself. Hence every continuous map preserves upwards directed suprema and downwards directed infima. Conversely, let $\mathfrak{F}$ be an ultrafilter on L and let $\mathfrak{F}' = \varphi(\mathfrak{F})$. Then the proof of (4.5) shows that $\lim \varphi(\mathfrak{F}) = \limsup \varphi(\mathfrak{F})$ $\leq \varphi(\limsup \mathfrak{F}) = \varphi(\lim \mathfrak{F}) = \varphi(\liminf \mathfrak{F}) \leq \liminf \varphi(\mathfrak{F}) = \lim \varphi(\mathfrak{F})$, i.e. $\lim \varphi(\mathfrak{F}) = \varphi(\lim \mathfrak{F})$.
This is equivalent to the continuity of φ. ∎

Corollary (4.6) gives raise to the question whether or not every complete lattice with the property that the IV-topology is T_2 is a quotient of a completely distributive lattice under a mapping preserving upwards directed suprema and downwards directed infima, i.e. a continuous and monoton mapping. We shall prove later on that this is a least true for meet continuous lattices and we will see in a moment that this is true for distributive lattices. But we do not know the answer in general.

Let us firstly recall some facts from the Priestley duality:

4.8 THEOREM [B]: Let $\underline{LD}$ the category of all distributive lattices with 0 and 1 together with all lattice homomorphisms preserving 0 and 1 as morphisms and let $\underline{PZ}$ be the category of all 0-dimensional compact (T_2) ordered spaces satisfying $x \not\leq y \Rightarrow \exists\, U = \uparrow U$ clopen, $x \in U$, $y \notin U$, together with all continuous order preserving mappings as morphisms. Then $\underline{LD}$ and $\underline{PZ}$ are dually equivalent under the contravariant functors $\underline{LD}(-,2): \underline{LD} \to \underline{PZ}$, where $\underline{LD}(L,2)$ is equipped with the pointwise ordering and the topology induced from the product topology of discret space 2, and $\underline{PZ}(-,2): \underline{PZ} \to \underline{LD}$, where the lattice operations $\wedge$ and $\vee$ on $\underline{PZ}(P,2)$ are defined pointwise. The natural transformations $\eta: 1\!\!1_{\underline{LD}} \to \underline{PZ}(\underline{LD}(-,2),2)$ and $\varepsilon: 1\!\!1_{\underline{PZ}} \to \underline{LD}(\underline{PZ}(-,2),2)$ are given by

$$\eta_L : L \to \underline{PZ}(\underline{LD}(-,2),2)$$
$$a \mapsto \hat{a}, \quad \hat{a}(\varphi) = \varphi(a)$$

and

$$\varepsilon_P : P \to \underline{LD}(\underline{PZ}(P,2),2)$$
$$p \mapsto \hat{p}, \quad \hat{p}(\varphi) = \varphi(a). \quad \blacksquare$$

As a consequence we have

4.9 PROPOSITION: Let X be a partially ordered set. Then the free distributive lattice generated by X is isomorphic to $\underline{PZ}(\tilde{X},2)$, where $\tilde{X}$ denotes the set of all order preserving mappings $\varphi: X \to 2$, equipped with the topology of pointwise convergence.

Proof. Clearly, $\tilde{X}$ is a zero-dimensional compact space. Moreover, if L is the free distributive lattice generated by X, then $\underline{LD}(L,2) \cong \{\varphi \mid \varphi: X \to 2 \text{ order preserving}\} = \tilde{X}$, hence $L \cong \underline{PZ}(\underline{LD}(L,2),2) \cong \underline{PZ}(\tilde{X},2)$. $\blacksquare$

In the sequel, for a partially ordered set X the continuous lattice $\tilde{X}$ is always endowed with the CL-topology.

4.10 PROPOSITION. Let P be a $\underline{PZ}$-object and $\varphi : P \to 2$ be a monoton mapping. Then $\varphi = \sup\{\inf\{\psi : \psi \in \underline{PZ}(P,2),\ \psi \geq \chi\} : \chi \leq \varphi,\ \chi^{-1}(1) \text{ closed}\} =$

$= \inf\{\sup\{\psi : \psi \in \underline{PZ}(P,2);\ \psi \leq \chi\} : \chi \geq \varphi,\ \chi^{-1}(1) \text{ open}\}$.

Furthermore, all occuring suprema and infima are directed.

Proof. First, let $\chi : P \to 2$ be monoton such that $\chi^{-1}(1)$ is closed. Then $\chi^{-1}(1) = \bigcap\downarrow\{U : \chi^{-1}(1) \subseteq U = \uparrow U,\ U \text{ clopen}\}$ by the definition of $\underline{PZ}$-objects and an easy compactness argument. Hence $\chi = \inf\{\psi_U : \chi^{-1}(1) \subseteq U = \uparrow U,\ U \text{ clopen}\} = \inf\{\psi : \psi \in \underline{PZ}(P,2),\ \psi \geq \chi\}$, where ψ_U denotes the caracteristic function of U, and this infimum is updirected. Next, let $\varphi : P \to 2$ be monoton. Then $\varphi^{-1}(1) = \bigcup\uparrow\{A,\ \varphi^{-1}(1) \supseteq A = \uparrow A,\ A \text{ closed}\}$. Hence $\varphi = \sup\{\chi_A : \varphi^{-1}(1) \supseteq A = \uparrow A,\ A \text{ closed}\} = \sup\{\chi : \chi \leq \varphi,\ \chi^{-1}(1) \text{ closed}\}$, where again χ_A denotes the characteristic function of A, and this supremum is also directed. This proves the first equality. The proof for the second equality goes similarly. ∎

Now let us return to generalized continuous lattices:

4.11 PROPOSITION. Let L be a gCL and $(I, \leq)$ a downwards directed index set. For each $i \in I$ let $D_i \subseteq L$ be an upwards directed subset such that $i \leq j$ implies $D_i \subseteq D_j$. Then $\bigwedge_{i \in I} \bigvee D_i = \bigvee_{d \in \prod_{i \in I} D_i} \bigwedge_{i \in I} d(i)$.

PROOF. The inequality „$\geq$" holds in every complete lattice. Conversely, let $F \subseteq L$ be a finite subset that guards $\bigwedge_{i \in I} \bigvee D_i$ from below. Then for every $i \in I$ there is a $d \in D_i$ such that $d \in \uparrow F$. Hence we can find an $f \in F$ such that $\{i \in I \mid \exists d \in D_i\ f \leq d\}$ is cofinal in I. This certainly implies $\{i \in I \mid \exists d \in D_i : f \leq d\} = I$. Hence there is an $d \in \prod_{i \in I} D_i$ such that $\bigwedge_{i \in I} d(i) \geq f \in \uparrow F$. ∎

4.12 PROPOSITION. Let L be a $\underline{gCL}$ and let $\tilde{L}$ denote the $\underline{PZ}$-object of all monoton mappings $\varphi : L \to 2$. Further, let $\lambda : \underline{PZ}(\tilde{L}, 2) \to L$ be an $\wedge$-homomorphism and let $\tilde{\tilde{L}}$ be the set of all monoton mappings $\varphi : \tilde{L} \to 2$. Then the mapping

$$\underline{\lambda} : \tilde{L} \to L, \quad \underline{\lambda}(\varphi) := \sup\{\inf\{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{L},2),\ \psi \geq \chi\} : \chi \leq \varphi,\ \chi^{-1}(1) \text{ closed}\}$$

preserves down directed infima.

Dually, if L^{op} is a $\underline{gCL}$ and if $\lambda : \underline{PZ}(\tilde{L}, 2) \to L$ is a $\vee$-homomorphism, then

$$\overline{\lambda} : \tilde{\tilde{L}} \to L, \quad \overline{\lambda}(\varphi) := \inf\{\sup\{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{L}, 2);\ \psi \leq \chi\} : \chi \geq \varphi;\ \chi^{-1}(1) \text{ open}\}.$$

Proof. Let $I \subseteq \tilde{\tilde{L}}$ be downwards directed and for every $\varphi \in I$ let $D_\varphi = \{\inf\{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{L},2), \psi \geq \chi\} : \chi \leq \varphi,\ \chi^{-1}(1) \text{ closed}\}$. Then $\sup D_\varphi = \underline{\lambda}(\varphi)$ and $\varphi \leq \varphi'$ implies $D_\varphi \subseteq D_{\varphi'}$. Hence we have $\inf\{\underline{\lambda}(\varphi) : \varphi \in I\} = \bigwedge_{\varphi \in I} \bigvee D_\varphi = \bigvee_{d \in \prod D_\varphi} \bigwedge_{\varphi \in I} d(\varphi)$ by (4.11). We want to prove that

$\bigvee_{\alpha \in \prod_{\varphi\in I} D_\varphi} \bigwedge_{\varphi \in I} \alpha(\varphi) \leq \underline{\lambda}(\inf I)$. Let $\alpha \in \prod_{\varphi \in I} D_\varphi$. Then we have to show that $\bigwedge_{\varphi\in I} \alpha(\varphi) \leq \underline{\lambda}(\inf I) = \sup\{\inf\{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{\Sigma},2),\ \psi \geq \chi\} : \chi \leq \inf I,\ \chi^{-1}(1) \text{ closed}\}$.

Now $\alpha(\varphi) = \inf\{\lambda(\psi) ; \psi \in \underline{PZ}(\tilde{\Sigma},2),\ \psi \geq \chi_\varphi\}$ for some $\chi_\varphi \leq \varphi$, $\chi_\varphi^{-1}(1)$ closed. Choose $\chi_0 := \inf\{\chi_\varphi \mid \varphi \in I\}$. Then $\chi_0 \leq \inf I$ and $\chi_0^{-1}(1) = \bigcap_{\varphi\in I} \chi_\varphi^{-1}(1)$ is closed. Hence it is enough to show that $\bigwedge_{\varphi \in I} \alpha(\varphi) \leq \lambda(\psi)$ for every $\psi \in \underline{PZ}(\tilde{\Sigma},2)$, $\psi \geq \chi_0$. But for every $\psi \in \underline{PZ}(\tilde{\Sigma},2)$, $\psi^{-1}(1)$ is clopen and $\psi^{-1}(1) \supseteq \chi_0^{-1}(1) = \bigcap_{\varphi\in I} \chi_\varphi^{-1}(1)$. Therefore we can find $\varphi_1, \dots, \varphi_m \in I$ such that $\chi_{\varphi_1}^{-1}(1) \cap \dots \cap \chi_{\varphi_m}^{-1}(1) \subseteq \psi^{-1}(1)$, using the compactness of $\tilde{\Sigma}$. Again, the compactness of $\tilde{\Sigma}$ yields elements $\psi_1, \dots, \psi_n \in \underline{PZ}(\tilde{\Sigma},2)$ such that $\chi_{\varphi_i} \leq \psi_i$ and $\psi_1 \wedge \dots \wedge \psi_n \leq \psi$. But now the fact that λ is an $\wedge$-homomorphism allows us to conclude $\bigwedge_{\varphi\in I} \alpha(\varphi) \leq \lambda(\psi_1) \wedge \dots \wedge \lambda(\psi_n) = \lambda(\psi_1 \wedge \dots \wedge \psi_n) \leq \lambda(\psi)$.

This proves $\inf \underline{\lambda}(I) \leq \underline{\lambda}(\inf I)$. The other inequality is always true. – The second claim can be proven dually. ∎

4.13 PROPOSITION. If L and L^{op} are gCL and if $\lambda : \underline{PZ}(\tilde{\Sigma},2) \to L$ is a lattice homomorphism, then $\underline{\lambda} : \tilde{\Sigma} \to L$ and $\bar{\lambda} : \tilde{\Sigma} \to L$ agree.

Proof. We first show that for every $\varphi \in \tilde{\Sigma}$ the inequality $\underline{\lambda}(\varphi) \leq \bar{\lambda}(\varphi)$. This inequality is true for every complete lattice L and every monotone map $\lambda : \underline{PZ}(\tilde{\Sigma},2) \to L$: Let $\chi \leq \varphi \leq \gamma$ such that $\chi^{-1}(1)$ is closed and $\gamma^{-1}(1)$ is open. Then we have to show that $\inf\{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{\Sigma},2),\ \chi \leq \psi\} \leq \sup\{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{\Sigma},2),\ \psi \leq \gamma\}$. $\chi \leq \gamma$ implies $\chi^{-1}(1) \subseteq \gamma^{-1}(1)$. Now the compactness of $\tilde{\Sigma}$ and the fact that $\tilde{\Sigma}$ has enough clopen uppersets yield an open and closed upperset $\mathcal{U}$ such that $\chi^{-1}(1) \subseteq \mathcal{U} \subseteq \gamma^{-1}(1)$. Now the characteristic function $\psi_{\mathcal{U}}$ of $\mathcal{U}$ satisfies $\chi \leq \psi_{\mathcal{U}} \leq \gamma$ and hence $\inf\{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{\Sigma},2),\ \chi \leq \psi\} \leq \lambda(\psi_{\mathcal{U}}) \leq \sup\{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{\Sigma},2),\ \psi \leq \gamma\}$.

To show the converse inequality, let $I := \{\chi \in \tilde{\Sigma} : \chi \leq \varphi,\ \chi^{-1}(1) \text{ is closed}\}$, $J := \{\gamma \in \tilde{\Sigma},\ \varphi \leq \gamma,\ \gamma^{-1}(1) \text{ is open}\}$. For every $\chi \in I$ let $D_\chi := \{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{\Sigma},2),\ \chi \leq \psi\}$ and for every $\gamma \in J$ let $D_\gamma = \{\lambda(\psi) : \psi \in \underline{PZ}(\tilde{\Sigma},2),\ \psi \leq \gamma\}$. Then I is updirected, J is downwards directed and for $\chi, \chi' \in I$, $\chi \leq \chi'$ implies $D_\chi \subseteq D_{\chi'}$ as well as for $\gamma, \gamma' \in I$, $\gamma \leq \gamma'$ implies $D_\gamma \subseteq D_{\gamma'}$. Moreover, by (4.11) we have $\underline{\lambda}(\varphi) = \bigvee_{\chi\in I} \bigwedge D_\chi = \bigwedge_{\alpha \in \prod D_\chi} \bigvee_{\chi \in I} \alpha(\chi)$ and $\bar{\lambda}(\varphi) = \bigwedge_{\gamma\in J} \bigvee D_\gamma = \bigvee_{\beta \in \prod D_\gamma} \bigwedge_{\gamma\in J} \beta(\gamma)$.

Fix $\alpha \in \prod D_\chi$ and $\beta \in \prod D_\gamma$. Then we have to show that $\bigwedge_{\gamma \in J} \beta(\gamma) \leq \bigvee_{\chi\in I} \alpha(\chi)$. But for every $\gamma \in J$ we have $\beta(\gamma) = \lambda(\psi_\gamma)$ for some $\gamma \geq \psi_\gamma \in \underline{PZ}(\tilde{\Sigma},2)$ and for every $\chi \in I$ we have $\alpha(\chi) = \lambda(\psi_\chi)$ for some $\chi \leq \psi_\chi \in \underline{PZ}(\tilde{\Sigma},2)$. Moreover, $\bigcap\{\psi_\gamma^{-1}(1) \mid \gamma \in J\} \subseteq \bigcap\{\gamma^{-1}(1) : \gamma \in J\} = \varphi^{-1}(1) = \bigcup\{\chi^{-1}(1) : \chi \in I\} \subseteq \bigcup\{\psi_\chi^{-1}(1) : \chi \in I\}$. Because $\psi_\gamma^{-1}(1)$ and $\psi_\chi^{-1}(1)$ are clopen for $\gamma \in J$, $\chi \in I$ and because $\tilde{\Sigma}$ is compact, we can find $\gamma_1, \dots, \gamma_n \in J$ and $\chi_1, \dots, \chi_m \in I$ such that $\psi_{\gamma_1}^{-1}(1) \cap \dots \cap \psi_{\gamma_n}^{-1}(1) \subseteq \psi_{\chi_1}^{-1}(1) \cup \dots \cup \psi_{\chi_m}^{-1}(1)$, i.e. $\psi_{\gamma_1} \wedge \dots \wedge \psi_{\gamma_n} \leq \psi_{\chi_1} \vee \dots \vee \psi_{\chi_m}$. Now the fact that λ is a lattice homomorphism yields $\bigwedge_{\gamma\in J} \beta(\gamma) \leq \beta(\gamma_1) \wedge \dots \wedge \beta(\gamma_n) = \lambda(\psi_{\gamma_1}) \wedge \dots \wedge \lambda(\psi_{\gamma_n}) = \lambda(\psi_{\gamma_1} \wedge \dots \wedge \psi_{\gamma_n}) \leq \lambda(\psi_{\chi_1} \vee \dots \vee \psi_{\chi_m}) = \lambda(\psi_{\chi_1}) \vee \dots \vee \lambda(\psi_{\chi_m}) = \beta(\chi_1) \vee \dots \vee \beta(\chi_m) \leq \bigvee_{\chi \in I} \beta(\chi)$. ∎

Now we can prove a partial reverse of Corollary (4.6):

4.14 THEOREM. Let L be a distributive lattice such that L and L^{op} are gCL. Then

(i) There exists a completely distributive lattice L' and a surjective mapping $\varphi: L' \to L$ preserving upwards directed suprema and downwards directed infima. Moreover, L' can be choosen to be $\tilde{\mathcal{I}}$ and $\varphi: \tilde{\mathcal{I}} \to L$ to be an extension of the canonical map $\lambda: PL(\mathcal{I}, 2) \to L$, which lifts the identity $id: L \to L$ (Recall that $PL(\mathcal{I},2)$ is the free distributive lattice generated by L). In this case, φ is a lattice homomorphism on a dense sublattice of L'.

(ii) The CL-topologies on L and L^{op} and the intervall topology on L agree. Especially, the intervall topology on L is T_2.

Proof. (i) Let $L' = \tilde{\mathcal{I}}$ and $\lambda: \underline{PL}(\mathcal{I}, 2) \to L$ the canonical map which lifts $id: L \to L$. Then λ is a lattice homomorphism. Let $\varphi = \underline{\lambda} = \overline{\lambda}$. Then for $\psi_0 \in PL(\mathcal{I}, 2)$ we have $\varphi(\psi_0) = \sup\{\inf\{\lambda(\psi): \chi \leq \psi \in \underline{PL}(\mathcal{I},2)\} : \chi \leq \psi_0, \chi^{-1}(1) \text{ closed}\} = \inf\{\lambda(\psi): \psi_0 \leq \psi \in \underline{PL}(\mathcal{I},2)\} = \lambda(\psi_0)$, hence φ extends λ. Moreover, φ preserves upwards directed suprema and downwards directed infima by (4.12). Further, $PL(\mathcal{I},2)$ is dense in $\tilde{\mathcal{I}}$ (4.10). This proves (i). Because $\tilde{\mathcal{I}}$ is a completely distributive lattice, (ii) is an immediate consequence of (4.6) and (4.2). ∎

4.15 COROLLARY. Let L be an distributive lattice. TAE

(i) The IV-topology on L is T_2.

(ii) L is a quotient of a completely distributive lattice under a mapping preserving directed suprema and infima. ∎

5. Characterization of Generalized Continuous Lattices by the Lattice of Scott Open Subsets.

It occurs all the time that every property of a complete lattice L can be characterized by a stronger property of the lattice $O(L)$ of Scott open subsets of L. In this section we want to draw our attention to this phenomenon.

Let L be a complete lattice. With $O(L)$ (resp. $C(L)$) we denote the lattice of all Scott open (resp. closed) subsets of L and with $U(L)$ (resp. $D(L)$) we denote the (completely distributive) lattice of all upper sets (resp. downsets) of L. Then we have a mapping

$$^{\circ d}: U(L) \to O(L)$$
$$A \mapsto \overset{\circ d}{A}$$

where $\overset{\circ d}{A}$ denotes the largest Scott open subset contained in A. $^{\circ d}$ preserves arbitrary infima. Dually, we have a mapping

$$^{-d}: D(L) \to C(L)$$
$$A \mapsto \bar{A}^d$$

where $\bar{A}^d$ is the smallest Scott closed subset containing A. Clearly, $^{-d}$ preserves arbitrary suprema. Note that $^{\circ d}: U(L) \to O(L)$ (resp. $^{-d}: D(L) \to C(L)$) is the left adjoint (resp. right adjoint) to the inclusion map.

5.1 Theorem. Let L be a complete lattice. TAE:

(i) L is a generalized continuous lattice.

(ii) $^{\circ d}: U(L) \to O(L)$ preserves upwards directed suprema.

(ii') $^{-d}: D(L) \to C(L)$ preserves downdirected infima.

(iii) $(O(L), \cap)$ is a continuous lattice and the CL-topology agrees with the IV-topology, i.e. the IV-topology is T_2.

(iii') $(C(L), \cup)$ is a continuous lattice and the CL-topology agrees with the IV-topology.

Proof. Clearly (ii) and (ii') as well as (iii) and (iii') are equivalent.

(i) ⇒ (ii'): For $A \subseteq L$ let $\bar{A}$ be the closure in the CL-topology. Then for $A \in D(L)$ we have $\bar{A}^d = \downarrow \bar{A}$ by Proposition 2.9. Let $\{A_i : i \in I\}$ be a downdirected net of downsets and let $a \in \bigcap \bar{A}_i^d$. Then for every Scott open neighborhood $\mathcal{U}$ of a we have $\mathcal{U} \cap A_i \neq \emptyset$. Because $\{A_i : i \in I\}$ is downdirected, the set $\{\mathcal{U} \cap A_i : i \in I,\ a \in \mathcal{U}$ Scott open$\}$ forms a filter base. Let $\mathfrak{F}$ be an ultrafilter containing this base. Then $A_i \in \mathfrak{F}$ for every $i \in I$ and hence for every $M \in \mathfrak{F}$ we have $A_i \cap M \neq \emptyset$. This implies $\bigwedge M \leq \bigwedge (A_i \cap M) \in A_i$, i.e. $\bigwedge M \in A_i$ for every $M \in \mathfrak{F}$ and every $i \in I$. But this means

$\bigwedge M \in \bigcap \downarrow A_i$ for every $M \in \mathfrak{F}$. Therefore we can conclude $\lim \mathfrak{F} = \liminf \mathfrak{F} \in \overline{\bigcap \downarrow A_i}^{d}$. On the other hand, $\mathfrak{F}$ contains all Scott open neighborhoods of a. If $\lim \mathfrak{F} \geq a$ would not be true, then we could find a finite subset $F \ll a$ with $\downarrow \lim \mathfrak{F} \cap F = \emptyset$. Because $\uparrow F = \bigcup \{\uparrow f : f \in F\} \in \mathfrak{F}$, we can find an $f \in \mathfrak{F}$ such that $\uparrow f \in \mathfrak{F}$. But then $\liminf \mathfrak{F} = \lim \mathfrak{F} \geq f$, a contradiction. Hence $a \leq \lim \mathfrak{F} \in \overline{\bigcap \downarrow A_i}^{d}$. This proves $\bigcap \downarrow \overline{A_i}^{d} \subseteq \overline{\bigcap \downarrow A_i}^{d}$. The other inclusion is obvious.

(ii') $\Rightarrow$ (iii') follows from (5.1) and (5.3), if we let $L' = D(L)$ and $\varphi = {}^{-d}$.

(iii') $\Rightarrow$ (i). The mapping $a \mapsto \downarrow a : L \to C(L)$ preserves arbitrary infima and updirected suprema and $(C(L), \cap)$ is a gCL by (4.3). Now the result follows from (3.7) ■

6. CL-Quotients of Completely Distributive Lattices

Before we state the main theorem, let us give two definitions.

6.1 Definition. Let L be a complete lattice. Then the Bi-Scott topology on L is the topology generated by the Scott open sets of L and L^{op}. ■

6.2 Definition. Let L be a complete lattice and $\mathcal{U} \subseteq L$ be a Scott open filter. Then $\mathrm{Spec}(\mathcal{U}) := \{y : y \text{ is maximal in } L \setminus \mathcal{U}\}$. If $k \in L$ is a compact element, we set $\mathrm{Spec}(k) = \mathrm{Spec}(\uparrow k)$. ■

Note that for every Scott open filter $\mathcal{U}$ we have $\mathrm{Spec}(\mathcal{U}) \subseteq \mathrm{Irr}(L)$.

6.3 Theorem. Let L be a complete lattice. TAE:

(i) L is a CL-quotient of a completely distributive lattice.

(ii) L is meet continuous and the IV-topology and the CL-topology agree.

(iii) L is meet continuous and the IV-topology is T_2.

(iv) L is a continuous lattice, L^{op} is a generalized continuous lattice and the Bi-Scott topology agrees with the CL-topology.

(v) L is a continuous lattice and for open downsets $\mathcal{U}, \mathcal{V} \subseteq L$ we have $\overline{\mathcal{U}} \subseteq \mathcal{V}$ iff we can find $a_1, \dots, a_n \in \mathcal{V}$ such that $\mathcal{U} \subseteq \downarrow a_1 \cup \dots \cup \downarrow a_n$.

(vi) L is a continuous lattice and for Scott-open filter $F_1, F_2 \subseteq L$ with $\overline{F_1} \subseteq F_2$ there exists a finite subset $A \subseteq \mathrm{Spec}\, F_1$ such that $\mathrm{Spec}(F_2) \subseteq \downarrow A$.

(vii) L is a meet continuous bi-generalized continuous lattice.

Proof. (i) ⇒ (ii) is clear by (5.6) and (4.3).

(ii) ⇒ (iii) is easy.

(iii) ⇒ (iv) If the IV-topology is T_2, then L and L^{op} are gCL by (4.3). Moreover, every meet continuous gCL is a continuous lattice by (2.6), hence L is a continuous lattice. Moreover, it follows from (5.3) that the Bi-Scott topology is coarser than the CL-topology. A standard argument using the way below relation shows that the Bi-Scott topology is T_2.

(iv) ⇒ (v) If the Bi-Scott agrees with the CL-topology, then the CL-topology on L^{op} is coarser than the CL-topology on L. If L^{op} is a gCL, then the CL-topology on L^{op} is T_2, hence the CL-topologies on L and L^{op} will agree. This implies that the open downsets of L in the CL-topology are exactly the Scott open set of L^{op}. Furthermore $\overline{\mathcal{U}} \subseteq \mathcal{V}$ is equivalent to $\mathcal{U} \ll \mathcal{V}$ in $O(L^{op})$. Because $^{od}: \mathcal{U}(L^{op}) \to O(L^{op})$ is a CL-morphism by (5.1), its right adjoint $i: O(L^{op}) \to \mathcal{U}(L^{op})$, which is given by the inclusion map, preserves the way below relation.

Hence $\overline{\mathcal{U}} \subseteq \mathcal{V}$ implies $\mathcal{U} \ll \mathcal{V}$ in $\mathcal{U}(L^{op})$. But this is the case iff $\mathcal{U} \subseteq \downarrow F$ for some finite subset $F \subseteq \mathcal{V}$.

(v) ⇒ (vi) Let F_1 and F_2 Scott open filters with $\overline{F_1} \subseteq F_2$ and set $\mathcal{U} = L \setminus \overline{F_2}$, $\mathcal{V} = L \setminus \overline{F_1}$. Then $F_2 \subseteq \overline{F_2}$, hence $\overline{\mathcal{U}} = (L \setminus \overline{F_2})^{-} = L \setminus \dot{F_2} \subseteq L \setminus F_2 \subseteq L \setminus \overline{F_1} = \mathcal{V}$. Therefore we can find finitely many points $a_1, \ldots, a_n \in L \setminus \overline{F_1}$ such that $\overline{\mathcal{U}} \subseteq \downarrow a_1 \cup \ldots \cup \downarrow a_n$. For each $a_i \in L \setminus \overline{F_1} \subseteq L \setminus F_1$ pick $a_i \leq b_i \in \operatorname{Spec} F_1$ by Zorn's lemma. Then $L \setminus F_2 \subseteq \downarrow b_1 \cup \ldots \cup \downarrow b_n$, i.e. $\operatorname{Spec} F_2 \subseteq \downarrow b_1 \cup \ldots \cup \downarrow b_n$.

(vi) ⇒ (vii) By the assumption (vi), L is clearly meet continuous and gCL. Moreover, if $b \not\leq a$, pick open filters F_1, F_2 such that $\overline{F_1} \subseteq F_2$; $b \in F_1$, $a \notin \overline{F_2}$. Then we can find a finite subset $A \subseteq \operatorname{Spec} F_1$ such that $\operatorname{Spec} F_2 \subseteq \downarrow A$. Clearly $\uparrow b \cap A = \emptyset$ and $L \setminus \overline{F_2} \subseteq L \setminus F_2 \subseteq \downarrow \operatorname{Spec} F_2 \subseteq \downarrow A$. But now $a \ll A$, because for a downdirected set D, $\inf D \leq a$ implies $\inf D = \lim D \in L \setminus \overline{F_2}$. Therefore $d \in L \setminus \overline{F_2} \subseteq \downarrow A$ for some $d \in D$. Hence L^{op} is a gCL. Finally, the CL-topologies on L and L^{op} agree. Let $\mathfrak{F}$ be an ultrafilter on L. Then $\liminf \mathfrak{F} = \limsup \mathfrak{F}$, because otherwise we would have $\liminf \mathfrak{F} \not\geq \limsup \mathfrak{F}$. Now the above proof yields open filters F_1, F_2 such that $\overline{F_1} \subseteq F_2$, $\limsup \mathfrak{F} \in F_1$, $\liminf \mathfrak{F} \notin \overline{F_2}$ and a finite subset $A \subseteq \operatorname{Spec} F_1$ with $L \setminus \overline{F_2} \subseteq \downarrow A$. As $\liminf \mathfrak{F} = \lim \mathfrak{F}$ in the CL-topology on L and as $\lim \mathfrak{F} \in L \setminus \overline{F_2}$, we have $L \setminus \overline{F_2} \in \mathfrak{F}$, hence $\downarrow A \in \mathfrak{F}$. Because a is finite, we can find an $a \in A$ such that $\downarrow a \in \mathfrak{F}$. Therefore $\limsup \mathfrak{F} \in \downarrow a$, a contradiction to $F_1 \cap \downarrow a = \emptyset$.

(vii) ⇒ (i) Let $L' = D(L)$ and $\varphi : L' \to L$ be defined by $\varphi(A) = \inf(L \setminus A)$. Then φ is surjective and preserves arbitrary infima. Moreover, $\inf(L \setminus A) = \inf(L \setminus \uparrow\overline{A})$, where $\overline{A}$ is the closure of A in the CL-topology. Because L is a continuous lattice by (2.6), the mapping $\mathcal{U} \mapsto \inf(L \setminus \mathcal{U})$ from the lattice of open downsets into L preserves upwards directed suprema by [ ]. Further, by (5.1) the mapping $\mathcal{U} \mapsto \mathcal{U}^{od} = L \setminus \uparrow\overline{(L \setminus \mathcal{U})}$ from the lattice of all downsets into the lattice of open downsets preserves upwards directed suprema. Because φ is the composition of these two mappings, φ will also preserve upwards directed suprema. This completes the proof. ∎

The following Corollary introduces a kind of algebraic lattices, which maybe are interesting from the point of view of universal algebra.

6.4 COROLLARY Let L be an algebraic lattice TAE:

(i) Conditions (i)–(vii) of (6.3) hold.

(ii) For every compact element $k \in L$ the set $\operatorname{Spec}(k)$ is finite.

Proof If (i) holds, (ii) follows easily from condition (vi) in (6.3). Conversely, under the assumption (ii), the proof of condition (vii) of (6.3) is an easy modification of the proof (vi) ⇒ (vii) in (6.3). ∎

Recall that a complete lattice L is called <u>bi-continuous</u>, if L and L^{op} are continuous lattices and if the CL-topologies on L and L^{op} coincide. The following corollary is now an easy consequence of (2.6) and (4.3):

4.5 COROLLARY. Let L be a complete meet & join continuous lattice. T.A.E.:

(i) L is bicontinuous.

(ii) The intervall topology on L is T_2.

(iii) L is a CL-quotient of a completely distributive lattice. ∎

Questions: All papers should end with questions. This encourages readership of the paper in hopes of eventually working on and solving the problems.

(1) What topological spaces are the gCL's when endowed with the Scott topology (recall what continuous lattices are the "absolute retracts" in the category of T_0 (or all) topological spaces by D. Scott's work)?

(2) Is there an equational description of gCL's?

(3) What is the gCL monad over sets?

(4) Some natural occuring examples of gCL's would be of great interest.