SEMINAR ON CONTINUITY IN SEMILATTICES(SCS)

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TOPIC Locally quasicompact sober spaces are Baire spaces

REFERENCES Lawmann I (K.H.Hofmann and J.D.Lawson ,Irreducibility, Semigroup F.13(76/77

Lawmann II (" " " , The spectral theory of continuous lattices, on the referee's table

LEMMA 1. Let L be a continuous lattice and V a Scott-open subset such that for some sequence $a_0 > a_1 > a_2 > \ldots$ one has

(HYP) $Va_n \subseteq V$ for $n = 0,1,\ldots$

Then for all $v \in V$ there is an open filter $U \subseteq V$ such that $va_n \in U$, n 0,1,..

(Remark. For the applications in this memo, v=1 would suffice.)

Proof. Let $v \in V$ be given and set $b_0 = 1$. Suppose that we found elements $b_k, k=0,\ldots,n$ such that

(i) $vb_k \in V$ for $k = 0,\ldots,n$,

(ii) $b_k \ll va_{k-1}b_{k-1}$ for $k=1,\ldots,n$.

(For n=0 ,nothing is assumed in lieu of (ii)). We construct b_{n+1}: By (i) we have $vb_n \in V$, hence by (HYP) we have (1) $va_nb_n \in V$. Let D denote the directed set $\Downarrow va_nb_n$. Then (2) $\sup D = va_nb_n$ since L is continuous. In particular, L is meet-continuous, and thus $\sup vD = v \sup = v^2a_nb_n = va_nb_n \in V$ by (2) and (1). As V is Scott-open, $vD \cap V \neq \emptyset$, i.e. there is a $b_{n+1} \in D$ with (3) $vb_{n+1} \in V$ and $b_{n+1} \in \Downarrow va_nb_n$, i.e. (4) $b_{n+1} \ll va_nb_n$. Then (3) and (4) make (i) and (ii) valid with n+1 in place of n. By induction we thus have (i) and (ii) for all k=0,1,.. (resp.,k=1,2,...). Then (ii) implies $b_{n+1} \ll va_nb_n \leq b_n$, hence

(5) $b_{n+1} \ll b_n$ for all n. By (i) we note $b_k \geq vb_k \in V$, whence

(6) $\uparrow b_n \in V$ for all n. By (ii) we note $va_n \geq va_nb_n \geq b_{n+1} \in V$, whence

(7) $va_n \in \uparrow b_{n+1}$ for all n. Now set $U = \bigcup_{n=0}^{\infty} \uparrow b_n$. Then U is a filte as an ascending union of filters. By (5) it is an open filter, and by (6) $U \subseteq V$. But (7) shows $va_n \in U$ for all n. ▯

Sei $b_n = v \wedge a_n$; $b_n \in V$ nach (HYP)

Wähle $c_1 \in V$, $c_1 \ll b_1$

Dann $c_1 \wedge a_2 \in V$ nach (HYP)

Wähle $c_2 \in V$, $c_2 \ll c_1 \wedge a_1$

usw

LEMMA 2. Let L be a continuous lattice and $A \subseteq L$ a countable set such that $a \in A$, $x \neq 0$ implies $xa \neq 0$. Then $\mathrm{IRR}\, L \subseteq \uparrow xA$ implies $x = 0$.

Proof. Fix $x \neq 0$ and write $A = \{x_0, x_1, \ldots\}$ and set $a_n = x_0 \ldots x_n$. Then $xa_n \neq 0$ for all n (by induction). Apply Lemma 1 with $V = L \setminus \{0\}$, $v = x$, and find an open filter U such that $0 \notin U$, $xa_n \in U$ for all n. Now let p be a maximal element in $L \setminus U$. By Lawmann I, $p \in \mathrm{IRR}\, L$. BUT $p \in L \setminus U \subseteq L \setminus \bigcup \uparrow xa_n \subseteq L \setminus \bigcup \uparrow xx_n = L \setminus \uparrow xA$. □

LEMMA 3. Let L be a continuous lattice and $A \subseteq L$. Then the following are equivalent: (1) $\uparrow A \cap \mathrm{Spec}\, L$ is nowhere dense in $\mathrm{Spec}\, L$.

(2) $\mathrm{Spec}\, L \nsubseteq \uparrow x \cup \uparrow A$ for all x with $\mathrm{Spec}\, L \nsubseteq \uparrow x$.

(3) $\mathrm{Spec}\, L \nsubseteq \uparrow xA$ for all x with $\mathrm{Spec}\, L \nsubseteq \uparrow x$.

If L is also distributive and $A = \{a\}$, then these are also equivalent to (4) $0 \neq xa$, for all $x \neq 0$.

Proof. Write $X = \mathrm{Spec}\, L$. Then $\uparrow A \cap X$ is nowhere dense in X iff every non-empty open set in X meets the complement $X \setminus \uparrow A$, and since the closed sets of X are precisely the sets $\uparrow x \cap X$, this is equivalent to saying that for all x with $X \setminus \uparrow x \neq \emptyset$ we have $\emptyset \neq (X \setminus \uparrow x) \cap (X \setminus \uparrow A) = X \setminus (\uparrow x \cup \uparrow A)$, which shows the equivalence of (1) and (2). But $p \in X \setminus (\uparrow x \cup \uparrow A)$ iff $x \nleq p$ and $a \nleq p$ for all $a \in A$ iff $xa \nleq p$ for all $a \in A$, since p is prime, and this means that $p \in X \setminus \uparrow xA$. Thus (2) and (3) are equivalent. If L is distributive, then $\mathrm{Spec}\, L \nsubseteq \uparrow x$ iff $x \neq 0$ by Lawmann I, and $\mathrm{Spec}\, L \nsubseteq \uparrow xa$ iff $xa \neq 0$. Thus (3) and (4) are equivalent. □

RECALL. A topological space X is a Baire space iff for all closed subsets $Y \neq \emptyset$ of X and each sequence $Y_1, Y_2, \ldots$ of subspaces of Y which are (closed and) nowhere dense in Y, we have $\bigcup Y_n \neq Y$. □

THEOREM 4. Every locally quasicompact sober space is a Baire space.

Proof. Let $X \neq \emptyset$ be locally quasicompact sober. By Lawmann II we may assume $X = \mathrm{Spec}\, L$ for a continuous Heyting algebra L. Each closed subset of X is of the form $\uparrow x \cap X = \uparrow x \cap \mathrm{IRR}\, L = \mathrm{IRR}\, \uparrow x = \mathrm{Spec}\, \uparrow x$, hence is itself the spectrum of a continuous Heyting algebra. Thus it suffices to show that for any sequence X_n of nowhere dense closed sets in X we have $X \neq \bigcup X_n$.