SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC Remark on Hofmann's SCS Memo 1/18/78

REFERENCE p. 1–3 of the above Memo

This is a direct proof of Hofmann's theorem: Every locally quasi-compact sober space is Baire.

(It may be that this is nothing ~~than~~ but Hofmann's original proof which I have retranslated from its abstract continuous lattice setting.)

LEMMA. If $\underline{F}$ is a filter which has a basis of open as well as a basis of quasicompact sets on a sober space, then $\underline{F}$ has a non-empty intersection.

Proof. As $\underline{F}$ has a basis of quasicompact sets, the union of every updirected family of open sets not belonging to $\underline{F}$ does not belong to $\underline{F}$ either. By Zorn's lemma, there is then an open set U which is maximal among the open sets not belonging to $\underline{F}$. Clearly, U cannot be the intersection of any two open sets containing U properly. Hence, the complement of U is an irreducible closed set. As the space is supposed to be sober, the complement of U is the closure of a point p. This p belongs to every open set not contained in U. Consequently p is in the intersection of $\underline{F}$, as $\underline{F}$ has a basis of open sets(which are not contained in U by the construction). §

Now, let Y be a locally quasicompact sober space. Let (U_n) be a sequence of dense open subsets of Y. We show that the intersection of the U_n is non-empty. For this, we may suppose that U_n contains U_{n+1} for all n. By induction, we construct a sequence of open sets V_n and a sequence of quasicompact sets Q_n such that $V_{n+1} \subset Q_{n+1} \subset U_{n+1} \cap V_n$: Let V_1 be any open set which is non-empty and contained in some quasicompact set $Q_1 \subseteq U_1$.

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(The existence of such things follows from the local quasicompactness.) As U_2 is dense, $V_1 \cap U_2$ is not empty, and we can find a non-empty open set V_2 contained in some quasicompact set $Q_2 \subseteq V_1 \cap U_2$, etc. Clearly the sequence V_n and Q_n generate the same filter. By the forgoing lemma, this intersection is not empty. As $V_n \subseteq U_{n-1}$ we have proved the assertion that the intersection of the U_n is not empty.
Now let X be a locally quasicompact sober space. Let U_n be a sequence of open subsets of X the intersection of which is also open. Then the complement Y of the intersection of the U_n is closed and, consequently, also locally quasicompact and sober. By the assertion proved in the previous paragraphe, not all of the sets $U_n \cap Y$ were dense in Y. Thus, we have proved that X is a Baire space, if we take as definition: A space X is a Baire space, if for no proper open subset U there is a sequence of open subsets U_n such that the intersection of the U_n is U and such that every $U_n \setminus U$ is dense in $X \setminus U$; this is equivalent to saying: no non-empty closed subset Y of X is the union of a sequence Y_n of closed subsets which are nowhere dense in Y.

REMARK. A little more abstract(and more general)version of the above lemma reads as follows:
Let X be a sober space and $\underline{F}$ a Scott open filter of the lattice $O(X)$ of open subsets of X. Then $\underline{F}$ has a non-empty intersection.
The proof remains essentially the same. The same proof as above then shows that every core compact sober space is Baire. (Unfortunately, Hofmann and Lawson have shown that such a space is locally quasicompact.)

COMMENT on the definition of a Baire space. The above definiton is not the one I am used to. The usual definition of a Baire space reads as follows: The intersection of a sequence of dense open subsets is dense or, equivalently, the union of a sequence of closed sets without interior points has no interior points. I can see that for regular spaces the above definition implies the usual one. The two definitions are not equivalent in general: With the above definition every closed subspace of a Baire space which is not true for Baire spaces according to the usual definition.
How about the following definition: X is Baire, if a closed subset with interior points cannot be the union of a sequence of closed subsets without interior points. (Then Hofmann's prop. 6 will not hold). But the theorem remains valid.
[illegible] ?

Now $X_n = X \cap \uparrow x_n$ with $xx_n \neq 0$ for all $x \neq 0$ by Lemma 3. Since L is distributive, IRR $L \setminus \{1\} = X$ (see Lawmann II). Apply Lemma 2 with $A = \{x_o, x_1, \ldots\}$ and find $X \nsubseteq \uparrow A = \bigcup \uparrow x_n$, i.e. $X \nsubseteq \bigcup X_n$. ∎

EXAMPLE 5. 1. Let $X = \mathbb{N}$ with upper sets open. Then X is locally quasicompact T_o, but is not a Baire space.

2. Let X be the set of all ordinals less than the first uncountable one with upper sets open. Then X is a locally quasicompact T_o Baire space which is not sober. ∎

Remark. Both examples are 1st countable. For the second, $\check{X}$ (the sobrification) is not.

PROPOSITION 6. Let L be a continuous Heyting algebra and X an order generating subspace of Spec L. (Note. By Lawmann II, <u>every</u> core compact space is of this form.) Now suppose that L satisfies the following countability hypothesis:

(COUNT) For each prime $p \neq 1$ the space $\uparrow p$ is first countable in $\not{p}L$ (w.r.t. the Lawson topology). ~~[illegible]~~ *Es wird benutzt: $\uparrow p \setminus \{p\}$ besitzt ko-initiale Folge.*

Then the following statements are equivalent:

(1) X is sober. (2) X = Spec L. (3) X is a Baire space.

Proof. (1) $\Leftrightarrow$ (2) by Lawmann II and (2) $\Rightarrow$ (3) by Theorem 4. Suppose not (2). Then we find a $p \in \text{Spec } L \setminus X$, and $Y = \uparrow p \cap X$ is a closed subspace of X. Note that by Lemma (3) we have

(i) for all ~~[illegible]~~ $v > p$, ($xv > p$ *for $x > p$* and thus) $\uparrow v \cap Y$ is nowhere dense closed in Y.

Let $V = \uparrow p \setminus \{p\}$; since p is prime, V is a filter. Since X is order generating, *(ii)* $p = \inf(\uparrow p \cap X) = \inf Y$ *whence $Y \neq \emptyset$.* By the definition of the Lawson topology, p now has a neighborhood basis in $\uparrow p$ of sets of the form $\uparrow p \setminus (\uparrow v_1 \cup \ldots \cup \uparrow v_n)$ with $v_k \in V$. Since ~~p is~~ prime, however we can say that (iii) in $\uparrow p$ the point p has a neighborhood basis of sets of the form $\uparrow p \setminus \uparrow v$, $v \in V$. By (COUNT) this becomes:

(iv) in $\uparrow p$ the point p has a basis of neighborhoods of the form $\uparrow p \setminus \uparrow v_n$ with a sequence $v_o \geq v_1 \geq \ldots$ in V.

~~The Lawson topology is Hausdorff,~~ whence $\bigcap(\uparrow p \setminus \uparrow v_n) = \{p\}$, and thus

(v) $V \subseteq \bigcup \uparrow v_n$. By (i), the closed sets $Y_n = \uparrow v_n \cap Y$ are nowhere dense in Y, but $\emptyset \neq Y = \bigcup Y_n$ by *(ii) and* (v). Thus X is not a Baire space, i.e. we proved not (3). ∎

Remark. The fact that X is order generating was *also* used in (1) $\Rightarrow$ (2).

Clearly (COUNT) is satisfied if L is metrizable (which is tantamount to saying that L contains a countable subset C such that $(\forall x,y)\ x \ll y \Rightarrow (\exists c)\ c \in C$ and $x \leq c \ll y$). Further, (COUNT) is equivalent to: "SpecL is first countable" (i.e. "$\check{X}$ is first countable"). This is satisfied if SpecL is second countable; but $O(\text{SpecL}) \cong O(X)$ (Lawmann II); hence the latter means that X is 2nd countable. ?

Consequence: For 2nd countable X, sobriety $\Longleftrightarrow$ Bairity.

Ist trotz allem richtig.

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