SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC antichains and equational compactness

REFERENCES 1 G. Gierz, J. Lawson ; generalized continous lattices SCS 2.9.77

2 G. Wenzel equational compactness in universal algebras , Manuskripte Universität Mannheim

3 D. Kelly ; a note on equationally compact algebras algebra universalis vol. 2 1972

The most exciting question in the theory of equational compactness is the problem of Mycielski :
"Are the equationally compact algebras in $\underline{K}$ the retracts of compact algebras in $\underline{K}$ "
For us , $\underline{K}$ is the class of lattices.
David Kelly has proved, that a complete meet and supcontinous lattice without infinite antichains is equationally compact.
But nothing was known about the topology of these lattices.
Fortunately, the theory of gCLs applies.

In the following, let L be a complete lattice without infinite antichain.

<u>1. Lemma</u> Each Scott-open set $U \subseteq L$ is of the form

$$U = L \setminus ({\downarrow}x_1 \cup {\downarrow}x_2 \cup \ldots \cup {\downarrow}x_n)$$

proof : Let U be a Scott-open set and $M \subseteq L$ be the set of all maximal elements of $L \setminus U$. As U is Scott-open, $L \setminus U = {\downarrow}M$. M is an antichain, because each element of M is maximal in $L \setminus U$ .Therefore, $M = \{x_1, x_2 \ldots x_n\}$ and $U = L \setminus {\downarrow}M = L \setminus ({\downarrow}x_1 \cup {\downarrow}x_2 \cup \ldots \cup {\downarrow}x_n)$.

<u>2.corollary</u> : In L , the Bi-Scott-topology coincides with the intervall-topology.

Now, I want to prove, that meet-continuity implies $L \in$ <u>CL</u> There should be a direct proof, but I did not yet succeed to find it.

<u>3.Lemma</u> Let $(y_j)_{j \in J}$ be an upward directed net, $\bigvee^{\uparrow}\{y_j \mid j \in J\} = y$ $\{x_1, x_2 \ldots x_n\}$ a maximal antichain in L , such that
$x_1, \ldots x_m < y$
$x_{m+1}, \ldots x_n \nleq y$
Then there is $j \in J$ and $i \in \{1, \ldots m\}$ such that $y_j \geqslant x_i$

proof : case 1 :There is some $i \in \{1, \ldots n\}$ and some $j \in J$ such that $y_j \geqslant x_i$. $y \geqslant y_j \geqslant x_i$ implies $i \in \{1, \ldots, m\}$
case 2 : $y_j \ngeqslant x_i$ for all $j \in J$ and all $i \in \{1, \ldots, n\}$
Define : $F_j := \{i \in \{1, \ldots, n\} \mid x_i \geqslant y_j\}$
(i) $F_j \neq \emptyset$ for all $j \in J$
$\{x_1, \ldots, x_n\}$ is a maximal antichain. so y_j is comparable with some x_i . According to the hypothesis $y_j \geqslant x_i$ does not hold. So $x_i \geqslant y_j$ is valid and $F_j \neq \emptyset$
(ii) $j \geqslant k$ implies $F_j \subseteq F_k$
So, it is shown : $\bigcap\{F_j \mid j \in J\} \neq \emptyset$
Now, let $i \in \bigcap\{F_j \mid j \in J\}$ Then $x_i \geqslant y_j$ holds for all $j \in J$
But this implies : $y = \sup\{y_j \mid j \in J\} \leqslant x_i$ ↯

4.LEMMA Let L be meet-continous, $y \in L$, $\{x_1, \dots x_n\}$ a maximal antichain, satisfying : $x_1, \dots x_m < y$

$x_{m+1} \dots x_n \nless y$

Then $\{x_1, \dots x_m\} \ll y$ is valid. (see [1] 1.1)

proof: Let $\sup\{w_j \mid j \in J\} \geq y$ for an upward directed net. Define $y_j := y \wedge w_j$ and apply Lemma 3 to $(y_j)_{j \in J}$

5.THEOREM A complete meet-continous lattice without infinite antichain is a continous lattice

proof : According to [1] 2.6 ; 1.3 I will try to prove (*)

(*) If $p \nleq q$, then there is a finite set F satisfying $F \ll p$ and $F \nleq q$

case 1 : $q \nleq p$ and there is $x < p$, $x \nleq q$

proof : Let $\{x_1, \dots x_n\}$ be a maximal antichain ,such that $x_1 = x$, $x_n = q$. Lemma 4 proves (*) , if one chooses F $F := \{x_1, \dots x_m\}$

case 2 : $q \nleq p$ and $x < p$ implies $x < q$

claim : p is compact and so $F := \{p\}$ satisfies (*)

proof : Let $(y_j)_{j \in J}$ be an upward directed net , such that $\sup\{y_j \mid j \in J\} \geq p$. We show, that there is some $j \in J$, such that $y_j \geq p$. Set $x_j := p \wedge y_j$ and we get $\sup\{x_j \mid j \in J\} = p$ Suppose, $x_j < p$ holds for each $j \in J$ .The "case 2 condition " implies $x_j \leq p \wedge q$ for all $j \in J$, which contradicts $\sup\{x_j \mid j \in J\} = p$. so there is some $j \in J$, such that $x_j \geq p$ and especially $y_j \geq p$.

case 3 : $p > q$ is a prime quotient.

proof : Let $\{x_1, \dots, x_n\}$ be a maximal antichain in L, such that $x_1 = q$, $x_1, \dots, x_m < p$, $x_{m+1}, \dots, x_n \nleq p$. our lemma 4 ensures us $\{x_1, \dots, x_m\} \ll p$. We will show now, that $\{p, x_2, \dots, x_m\}$ satisfies (*) . It is enouph, to prove the following claim.

claim : If $p \gg X$, $q \in X$ and p/q is prime, then $p \gg (X \setminus \{q\}) \cup \{p\}$

proof : Suppose , $\bigvee^{\uparrow}\{y_j \mid j \in J\} \geqslant p$. It is enouph, to show : If $y_j \geqslant q$ holds for any $j \in J$, then there is some $k \in J$, such that $y_k \geqslant p$. As $\sup\{y_j \mid j \in J\} \geqslant p$, there is some $k \in J$ satisfying $y_k > q$ and this implies $y_k \geqslant p$, because p/q is prime .

case 4 : There is some $x \in L$, satisfying $q < x < p$

proof : Let X be a maximal antichain satisfying $x \in X$. Again , lemma 4 proves the validity of $(\ast)$

<u>COROLLARY</u> <u>A meet and sup-continous complete lattice without infinite antichain is a bicontinous lattice</u>.

remark : It is possible, to avoid the theory of gCLs, if one proves $\bigwedge^{\downarrow}_{i \in I} \bigvee^{\uparrow}_{j \in J_i} x_j = \bigvee^{\uparrow}_{\prod_{i \in I} J_i} \bigwedge^{\downarrow}_{i \in I} x_{f(i)}$

but this proof is more complicated.

EQUATIONAL COMPACTNESS

<u>Definition</u> An algebra A is called eqationnally compact, if each system of equations has a solution in A , provided each finite subsystem has a solution.

Together with some familiar facts of the theory of equational compactness, the following theorem is just proved .
The results, which are needed, are :
" A compact algebra is equationally compact. "
proof : The solutions of the finite subsystems are a filter base and converge to the solution of the whole system.
" A retract B of an equational compact algebra A is e.c. ""
proof : Solve the equations in A and apply the retraction.
Grätzer, Lakser : " An equationally compact lattice is complete, meet and sup-continous. "

THEOREM : For a lattice L, not containing any infinite antichain, the following conditions are aquivalent :

a) L is equationally compact

b) L is complete, meetcontinous and supcontinous

c) L is a bicontinous lattice

d) L is a compact lattice

e) L is a retract of a compact lattice

THE IDEAL LATTICE

Proposition : Let H be a supsemilattice and $I(H)$ the ideal-lattice of H. The intervall-topology of $I(H)$ is T2 if and only if for each $p \in H$, there is no infinite set $X_p \subseteq H$, satisfying (1) and (2)

(1) $x \in X_p$ implies $x \not\geq p$

(2) $x, y \in X_p$, $x \neq y$ implies $x \vee y \geq p$

corollary If the supsemilattice H does not contain any infinite antichain, then the intervall-topology of $I(H)$ is T2 .

remark : In [1], 6.4, G. Gierz and J. Lawson have proved, that the intervall-topology of $I(H)$ is T2 iff for each $p \in H$, the set $\mathrm{spec}(p)$ of all maximal ideals not containing p is finite .

proof of the proposition : "$\Rightarrow$" Let X_p be an infinite set, satisfying the conditions (1) and (2) . For each $x \in X_p$ there is a maximal ideal, such that $x \in I_x$ but $p \notin I_x$. If $x \neq y \in X_p$ then $I_x \neq I_y$, because $y \in I_x$ implies $p \in I_x$. So, $\mathrm{spec}(p)$ is infinite , which contradicts the remark.

"$\Leftarrow$" Suppose, the intervall-topology of $I(H)$ is not T2 . According to the remark, there is some $p \in H$, such that the set $\mathrm{spec}(p)$ of maximal ideals not containing p is infinite.

Let $S_1 = \{I_n^1 \mid n \in \mathbb{N}\}$ be a countable subset of $\mathrm{spec}(p)$

Now, we want to construct an infinite set X_p satisfying (1) and (2), which will contradict our hypothesis.
As $I_1^1 = I_2^1$ are maximal with respect to $p \notin I$, there is some $x_1 \in I_1^1$ such that $x_1 \notin I_2^1$ and so, there is some $x_2 \in I_2^1$ such that $x_1 \vee x_2 \geqslant p$.
Set $X_2^1 := \{x_1, x_2\}$ and X_2^1 satisfies (1) and (2).
Suppose, X_n^1 is defined, such that (1) and (2) is valid and look at I_{n+1}^1. Either I_{n+1}^1 contains some $x \in X_n^1$, then define $X_{n+1}^1 := X_n^1$, or I_{n+1}^1 does not contain any $x \in X_n^1$.
Then there is some $x_{n+1} \in I_{n+1}^1$, such that $x_{n+1} \vee x \geqslant p$ for all $x \in X_n^1$. So $X_{n+1}^1 := X_n^1 \setminus \{x_{n+1}\}$ will satisfy (1) and (2)
Define : $X^1 := \bigcup \{ X_n^1 \mid n \in \mathbb{N}\}$
According to our hypothesis, X^1 satisfies (1) and (2) and is finite. X^1 has the additional property : For each $I \in S_1$ there is some $x \in X^1$, such that $x \in I$. So there is some $x^1 \in X^1$ such that
$S_2 = \{I \in S_1 \mid x^1 \in I\}$ is infinite. Define
$X_1^2 := X^1 \setminus \{x^1\}$, (which will satisfy (1) and (2)) and asume $S_2 = \{I_n^2 \mid n \in \mathbb{N}\}$.
If you apply the same construction to $S_2, S_3, \ldots\ldots$
you will get an ascending chain

$$X^1 \subsetneqq X^2 \subsetneqq X^3 \subsetneqq \ldots\ldots$$

such that each X_n satisfies the conditions (1) and (2).

$$X_p := \bigcup \{ X^n \mid n \in \mathbb{N} \}$$

now is infinite and still satifies (1) and (2), which is the wanted contradiction.