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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC: A review of a theorem of Dixmier's

REFERENCES : J.Dixmier, Sur les espaces localement quasi-compacts, Canadian J.Math. 20 (1968), 1093-1100.

SCS Compendium on Continuous Lattices, Darmstadt 1978.

In the paper mentioned in the references above, J.Dixmier discussed certain aspects of the general topology of locally quasi-compact spaces for the purpose of determining the purely topological nature of certain properties of the primitive ideal spectrum of a C* -algebra. Certain conclusions concern Borel sets (from which some exercises in the revision of the Compendium are drawn) and others pertain to the issue of Baire category (for which we now have better results, which also will be included in the Compendium (see also SCS Hofmann 1-18 -78 and SCS Keimel-Bauer 2-9-78)).

We review another topic which Dixmier discussed in his article. For reasons of the spectral theory of C*-algebras it is important to know the so-called Hausdorff-points of a space: A point is a Hausdorff point iff it can be separated by disjoint open sets from any point which is not in its closure. Dixmier uses for this purpose a concept which will will reintroduce in the form of "tied" elements in a lattice. These elements appear a bit artificial from a lattice theoretical point of view. And indeed a closer inspection from the vantage point of continuous lattices reveals that for all practical purposes in this context they can be replaced by pseudoprimes. In fact we prove here the following version of Dixmier's result:

Let X be a locally quasicompact T_0-space and consider it in the now standard manner as an order generating subspace of Spec L for a unique continuous Heyting-algebra L. Then the set of Hausdorff points in X is precisely the set of x in X which are minimal pseudoprimes.

Fred Watkins (whose dissertation concerns the application of lattice theoretical methods to the problem of closed two -sided prime ideals in C*-algebras -are they primitive?-) lectured on Dixmier's paper in the current graduate student seminar. In the course of these lectures we also discovered that the Lawson topology was introduced by Fell as early as 1961 on O(X) for locally quasicompact spaces X. The notes which follow emerged from the discussions around the seminar.

1) Notation. For any upper set X in a lattice L let $F(X) = \cup X^n$ denote the filter generated by X. □

2) DEFINITION . Let L be a (complete) lattice. We say that an element $x \in L$ is tied (lie in the sense of Dixmier) iff for any finite set $E \subsetneq L \setminus \downarrow x$ we have $\inf E \neq 0$. □ The set of all tied elements will be called $T(L)$.

Recall that an ultrafilter U in a lattice L is a maximal filter with $0 \notin U$.

3) LEMMA. In a (complete) lattice L, the following statements are equivalent for an element x in L:

(1) $x \in T(L)$

(2) $0 \notin F(L \setminus \downarrow x)$

(3) There is an ultrafilter U such that $U \cup \downarrow x = L$.

Proof. (1) <=> (2) is straightforward.

(2) => (3): By the axiom of choice (Zorn's Lemma), there is an ultrafilter U containing $F(L \setminus \downarrow x)$. It satisfies the requirement.

(3) $\Longrightarrow$ (2): From (3) we have $L \setminus \downarrow x \subseteq U$. Then $F(L \setminus \downarrow x) \subseteq U$, and so (2) follows. □

4) PROPOSITION . In any complete lattice L the set $T(L)$ is closed in the Lawson topology.

Proof. Suppose that $x \notin T(L)$. Then there is a finite set $E \subseteq L \setminus \downarrow x$ with $\inf F = 0$. Now $L \setminus \uparrow F \subseteq L \setminus T(L)$ and is $\omega(L)$-open, hence $\lambda(L)$-open, and contains x. □

We recall that the set of prime elements in a lattice is called PRIME L and the set of pseudoprimes ΨPRIME L (Compendium II-3.23). Further we recall from the work of Hofmann and Lawson on IRREDUCIBILITY and the SPECTRAL THEORY that for a continuous lattice L one has PRIME $L \subseteq \Psi$PRIME $L \subseteq$ WPRIME $L = \overline{\text{PRIME}}\ L$, and that for distributive continuous lattices (but not for others) equality holds between the last three. In the following we will refer to complete lattices in general and concentrate on primes and pseudoprimes only. Closures will always refer to the Lawson topology.

5) PROPOSITION . If L is a (complete) lattice, then $\Psi\mathrm{PRIME}\ L \subseteq T(L)$.
Proof. If p is a pseudoprime, then there is a prime ideal P with $p = \sup P$. Since P is a prime ideal, $L \setminus P$ is a filter containing any finite set E in the complement of $\downarrow p$. Hence $\inf E \in L \setminus P$, and so $\inf E \neq 0$ for such E. Hence $p \in T(L)$.

By Proposition 4 this says in particular that for any complete lattice we have $\overline{\mathrm{PRIME}\ L} \subseteq \overline{\Psi\mathrm{PRIME}\ L} \subseteq T(L)$.

6) PROPOSITION. Let L be a (complete) distributive lattice. Then every minimal element of $T(L)$ is pseudoprime.
Proof. Let p be a minimal element in $T(L)$. By 3.3) there is an ultrafilter U with $U \cup \downarrow p = L$. Since L is distributive, the complement $L \setminus U$ is a prime ideal (apply I-3.22 with $I = (0)$, $F = U$ and use the maximality of U). From $L \setminus U \subseteq \downarrow p$ we obtain $\sup(L \setminus U) \leq p$; ∴ the element $\sup(L \setminus U)$ is pseudoprime, hence tied by Proposition 5. Minimality of p then shows $p = \sup(L \setminus U)$. □

Notice that minimal elements in $T(L)$ exist on account of 4) if the graph of $\leq$ is closed in the Lawson topology. (By 7.14 of Chapter II this means that L is GCL.)

The next result gives the final reason why one might consider the somewhat artifical notion of tied elements; it also illustrates that in the case of a distributive continuous lattice the concept is superfluous, since the lattice theoretically more natural pseudoprimes serve the same purpose.

7) THEOREM. Let L be a continuous lattice and $X = \mathrm{Spec}\ L$ the space of its primes $p \neq 1$ with the hull kernel topology. Let p be a prime in X. We consider the following statements:

(1) p is minimal in $T(L)$.
(2) p is a Hausdorff point in X, i.e. for every $q \in X$ with $p \notin \overline{\{q\}}$ there are disjoint open neighborhoods of p and q in X, respectively.
(3) $(\forall q \in X)\ p \nleq q \Rightarrow (\exists u, v \in L)\ uv = 0$ and $u \nleq p$ and $v \nleq q$.
(4) The $\lambda(L)$-neighborhood filter of p on X agrees with the $\omega(L)$-neighborhood filter of p on X.
(5) The inclusion map $X \longrightarrow \Lambda L$ is continuous in p, where ΛL denotes the space L with the Lawson topology.
(6) p is minimal in $\overline{\mathrm{PRIME}}\ L$.

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6), and if L is distributive, then all of these conditions are equivalent.

Proof. The equivalences (2)<=>(3) and (4)<= >(5) are straightforward reformulations.

(1) => (3): Let p be minimal in T(L) and $p \nleq q$. Then $pq < p$ and so $pq \notin T(L)$, whence there is a finite set F in $L \setminus \downarrow pq$ with $\inf F = 0$. Set $u = \inf(F \setminus \downarrow p)$ and $v = \inf(F \cap \downarrow p)$. Then $uv = \inf F = 0$; secondly, $u \notin \downarrow p$, since p is prime, and thirdly $v \notin \downarrow q$, for otherwise $F \cap \downarrow pq = F \cap \downarrow p \cap \downarrow q = 0$.

(3) => (1): Assume (2) and suppose that (1) does not hold. Then there is an $a \in T(L)$ with $a < p$. We now use the fact ... that X is order generating in a continuous latt. and find a $q \geq a$ with $p \nleq q$. Then by (2) we find $u, v \in L$ with $uv = 0$, $u \nleq p$, $v \nleq q$. Then $u \nleq a$ and $v \nleq a$. Since a is tied, $uv = 0$, and this is a contradiction.

(1) => (4) : Let U be a Scott neighborhood of p. We must find an $x \notin \downarrow p$ so that $X \setminus \uparrow x \subseteq U$. If no such x exists, then the sets $S_x = (X \setminus \uparrow x) \cap (L \setminus U)$ are non-empty for all x in the filter $L \setminus \downarrow p$. The collection of all S_x, $\lambda(L)$-x, is then a filter-basis on the Lawson quasicompact set $L \setminus U$ and thus has a cluster point y in $\overline{X} \cap (L \setminus U)$. In particular $y \in T(L)$ by Propositions 5 and 4. By the minimality of p in T(L) we obtain $y \in L \setminus \downarrow p$. Now we use the hypothesis that L is continuous and find a $u \ll y$ with $u \nleq p$. Then the Scott-(hence Lawson-) open neighborhood $\Uparrow u$ of y does not meet $X \setminus \uparrow u$, hence does not meet S_u, and this is a contradiction.

(4) =>(6) : Let $a \in \overline{\mathrm{PRIME}\ L}$ with $a \leq p$. Let U be any Scott-open neighborhood of p. By (4) we find an $x \in L \setminus \downarrow p$ with $X \setminus \uparrow x \subseteq U$. Hence $a \in L \setminus \uparrow x$. Since $a \in \overline{X}$ we have $a = \lim p_j$ for a net p_j of primes in X, and we may assume that $p_j \in L \setminus \uparrow x$. But then even $p_j \in X \setminus \uparrow x \subseteq U$. Thus $a \in \overline{U}$. But U was arbitrary; if we now use again the continuity of L, we can conclude that $p \leq a$. (GCL would suffice at this point.) Thus we have $a = p$, i.e. p is minimal in $\overline{\mathrm{PRIME}\ L}$.

(6) => (1): Let p be minimal in $\overline{\mathrm{PRIME}\ L}$ and let $a \leq p$ be tied. If L is continuous then the graph of $\leq$ is closed, and thus we may assume that a is minimal in T(L). If L is distributive, then Proposition 6 applies and shows that a is pseudoprime. But in a continuous distributive lattice we have $\Psi\mathrm{PRIME}\ L = \overline{\mathrm{PRIME}\ L}$ (according to the SPECTRAL THEORY of Hofmann and Lawson). Thus $a \in \overline{\mathrm{PRIME}\ L}$, and then by the minimality of p we have $a = p$. □

Note that only the conclusion (1) =>(2) did not use continuity of L. The simplest formulation of this result emerges in the case of continuous distributive lattices, i.e. continuous Heyting algebras. We summarize:

8) COROLLARY. 1) Let L be a continuous Heyting algebra. Then

$$\min T(L) = \min \Psi\mathrm{PRIME}\ L$$

(where min A denotes the set of all minimal elements in A).

2) Let X be a locally quasicompact T_0-space. Embed X as an order generating subset of Spec L for a continuous Heyting algebra L. (This is always possible after Hofmann and Lawson.) Then the set of Hausdorff points of X is precisely the set $X \cap \min \Psi\mathrm{PRIME}\ L$. □

9) COROLLARY. Let X be a locally quasicompact sober space. Then a point x is a Hausdorff point iff its neighborhood filter is also its neighborhood filter relative to the patch topology. □

In the context of Corollary 8 one might be tempted to believe that minimal pseudoprimes in a continuous Heyting algebra are always prime. This is not the case as the following example shows which was provided by John Isbell:

10) EXAMPLE. (J. Isbell). Let X be the T_1 space obtained on $\mathbb{N} \cup \{a,b\}$ by taking as basic neighborhoods of a, resp. b all sets containing a (resp., b) and having finite complement; all elements in $\mathbb{N}$ are isolated.

Now we take $L = O(X)$ and consider $p = \mathbb{N} \in L$. Clearly p is not prime (as the meet of the two elements $X \setminus \{a\}$ and $X \setminus \{b\}$). But p is pseudoprime since it is the sup of any maximal ideal M of subsets of $\mathbb{N}$ which is not only prime in $2^{\mathbb{N}}$ but also in L: Indeed if $xy \in M$ with $x \not\leq p$ and $y \not\leq p$, then $xy \in M$ would imply that xy cannot be cofinite, while $x \not\leq p$ and $y \not\leq p$ would imply that it is. A simple consideration along similar lines shows that no proper subst of $\mathbb{N}$ can be a pseudoprime in L, whence p is minimal. □