NOTES ON CHAINS in CL-OBJECTS (from KHH) 4-19-76

REFERENCE: Letter from KHH to AS of ~~2~~-3-22-76 (not circulated)
Letter from DS to KHH of 3-30-76 (circulated)
Notes on Stralka's congruence extension theorem from K
Nov-Dec.75 (circulated)

Ever since DIMENSION RAISING one had worked with chains inside a CL-object to obtain chain quotients. An indication of a general construction was given in ATLAS which was rightly critizized by Dana Scott. Shortly before, Al Stralka had asked me whether in a connected CL-object S one could always find, for any $x \ll 1$ a morphism $f:S \longrightarrow I$ with $x =$ min $f^{-1}(1/2)$. I believe I gave an argument under the hypothesis that $\uparrow x$ has a countable neighborhood basis, and a counterexa in the absence of this hypothesis.

I would like to record some general remarks.

and $a<b$

0. Remark. If $a \ll b$ in a CL-object, then there is an s with $a \ll s \ll b$ and $a < s$.

1. Definition. A subset C in a CL-object S is a <u>strict chain</u> if $x,y \in C$ implies that $x \ll y$ or $x = y$ or $y \ll x$. □

All singletons are strict chains; $\{0,1\}$ is a strict chain. All chains contained in $K(S)$ are strict chains.

2. Remark. The collection of all strict chains is $\subseteq$ - inductive. Indeed if $\{C_j : j \in J\}$ is a $\subseteq$-totally ordered family of strict chains and $C = \bigcup_J C_j$, then C is a strict chain, since for any $x,y \in C$ there is a $j \in J$ with $x,y \in C_j$, whence $x \ll y$ or $x = y$ or $y \ll x$ since C_j is strict.

3. Proposition. Let C be a strict chain in a CL-object S. Then C is contained in at least one maximal strict chain M.

Proof. Remark 2 and Zorn's Lemma. □

A <u>gap</u> in a poset S is an interval $[a,b]$ with $[a,b] = \{a,b\}$.

4. Proposition. Let C be a maximal strict chain in a CL-object. Then

(i) $0 \in C$.

(ii) If $[a,b]_C$ is a gap in C, then $b \in K(S)$,

(iii) C is complete.

Then ~~there~~ by Remark 0 there is an $s \in S$ with $a \ll s \ll b$, $a < s$. By maximality of C we have $s=b$. Thus $b \ll b$, whence $b \in K(S)$, ~~ie $t \ll b$ for all $t \in {\downarrow}b$. Thus if m is a maximal element in ${\uparrow}a \cap ({\downarrow}b \setminus \{b\})$, then $a \not\ll m$, i.e. $a \notin ({\downarrow}m)_o$ (i.e. a is in a face of ${\downarrow}m$)~~.

(iii) Let $X \subseteq C$ and $x = \sup_S X$.

Case 1. $x \in C$. Good.

Case 2. $x \notin C$. Then for any $c \in C$, $c < x$ there is a $d \in C$ with $c < d \leq x$; since C is strict, we have $c \ll d$, whence $c <$ By the maximality of C there must then be a $y \in C$ with $x < y$ but $x \not\ll y$. If there were a $u \in C$ with $x \leq u < y$, then $u \ll y$ (C is strict) whence $x \ll y$ which is impossible. Thus $y = \min({\uparrow}x \cap C)$. But then $y = \sup_C X$.

(iv) is trivial.

5. Lemma. For a maximal/strict chain C in S define $\varphi: C \longrightarrow S$ by $\varphi(c) = \sup_S$ ~~for~~ $\{d \in C: d \ll c\}$. Then φ is an injective comorphism, with $\varphi(c) \leq c$ for all $c \in C$.

Proof. (a) If $a < b$ in C then $a \ll b$ since C is strict. Thus $\varphi(a) \leq a \leq \varphi(b)$. If $\varphi(a) = \varphi(b)$, then $a = \sup_S\{c \in C : c \ll b\}$ i.e. $[a,b]_C$ is a gap. Thus $b \in K(S)$ by 4 (ii), .But then $b \ll b$ whence $b = \varphi(b)$. This implies $a = b$, a contradiction. Thus φ is injective.

(b) If $a \ll_C b$, then (i) $a < b$ or (ii) $a=b \in K(C)$. In case (ii) we have $a=b \in K(S)$ by 4 (ii), whence $a \ll_S b$ and $\varphi(a)=a=b=\varphi(b)$. Thus $\varphi(a) \ll_S \varphi(b)$. Now suppose $a < b$. If $[a,b]_C$ is a gap, then $b \in K(S)$, whence $\varphi(a) \leq a \ll_S b = \varphi(b)$. If $[a,b]_C$ is not a gap, then there is a $u \in C$ with $a < u < b$. Then $\varphi(a) \leq a \ll_S u$ ~~xx~~ $\leq \varphi(b)$ (as $u \ll b$, since C is strict), whence $\varphi(a) \ll_S \varphi(b)$.

(c) If $x = \sup_C X$, then (i) $x = \max X$ or (ii) $x \notin X$. In case (i) we have $\varphi(x) = \max \varphi(X)$ since φ is monotone. In cas (ii) the relation $c \ll x$ in C implies $c < x$ (since $x \notin K(C)$, hence implies) the existence of a $u \in X$ with $c < u < x$, so $c \ll u \ll x$, since C is strict. So $\sup_S \varphi(X) = \sup_S\{\sup_S \{c \mid c \ll u\} \mid u \in X\} = \sup_S\{c \mid$ there is a $u \in X$ with $c \ll u$ $= \sup_S\{c \mid c \ll x\} = \varphi(x) = \varphi(\sup_C X)$. □

According to ATLAS duality, from Lemma 5 the function φ has a morphism $\psi: S \longrightarrow C$ as left adjoint which satisfies the following conditions:

(1) $\psi(s) \geq c$ iff $s \geq \varphi(c)$ for all $s \in S$, $c \in C$.

(2) $\psi(s) = \sup_C \varphi^{-1}(\downarrow s) = \sup_C\{c \in C: \varphi(c) \leq s\}$

$= \sup_C\{c \in C: d \ll c \Rightarrow d \leq s \text{ for all } d \in C\}$

(3) $c = \psi\varphi(c) \leq \psi(c)$ for all $c \in C$.

(4) $\varphi(c) = \inf \psi^{-1}(\uparrow c) = \min \psi^{-1}(c) =$

$= \min\{s \in S: \psi(s) = c\}$

Indeed, (1) is the definition of adjointness, (2) ~~xx~~ comes from the determination of a left adjoint in terms of its right adjoints and the definition of φ. (3) follows from the fact that φ is injective (Lemma 5 above and ATLAS 1.12) and from $\varphi(c) \leq c$ for all c. (4) arises from the determination of a right adjoint in terms of its left adjoint plus the fact that the left adjoint ψ is surjective.

6. Lemma. Under the hyptheses of Lemma 5 the following statements are equivalent for an element $c \in C$.

(i) $c = \varphi(c)$.

(ii) $c = \min\{s \in S: \varphi(s) = c\}$

(iii) c is not isolated from below in C <u>in the induced topology</u>, i.e. for all $s \ll c$, $s \in S$ there is a $d \ll c$ $d \in \uparrow s \cap C$.

Proof. (i)<=>(ii) from (4) above. (i) <=> (iii) from the definition of φ in Lemma 5.□

7. Definition. We (prehaps temporarily) call an element $s \in S$ accessible, if there is a strict chain C with $s = \sup_S C$.□

8. LEMMA. If s has a countable neighborhood basis, then

U_n is open and $\overline{U}_{n+1} \subseteq U_n$. Then $\{\inf U_n : n=1,\ldots,n\}$ is a strict chain C with $\sup C = s$.[]

Other points s without a countable basis for their neighborho may be accessible. E.g. <u>every</u> point of a chain is accessible. Thus if $S = [0, \mathrm{OMEGA}]$ with the first uncountable ordinal OMEGA, then OMEGA does not have a countable neighborhood basis, but is accessible.

Let $S = 2^X$ (or I^X) with an uncountable set X. Then 1 is not accessible, for $s = (u_x)_{x \in X} \ll 1$ implies that all but a finit number of the u_x are 0.

9.THEOREM. Let $S \in \underline{CL}$. Then every strict chain $C_0 \subseteq S$ (such as $C_0 = \{1\}$ or $=\{0\}$) is contained in a maximal strict chain C. Moreover, $C \in \underline{CL}$ and there is a surjective $\underline{CL}$-morphism $\psi : S \longrightarrow C$ whose right adjoint is given by $c \longmapsto \sup_S \{d \in C : d \ll c\}$. If $c \in S$ is accessible, then there is a maximal strict chain C with $c \in C$ such that $c = \min\{s \in S : \psi(s) = c\}$. If $c \in S$ has a countable neighborhood basis, then c is accessible.

Proof. By Proposition 3, C_0 is contained in a maximal strict chain C. By Proposition 4, $C \in \underline{CL}$. By Lemma 5 and the subsequent remarks, $\psi : S \longrightarrow C$ exists with the specified properties. If c is accessible, we have a strict chain C_0 with $\sup_S C_0 = c$. Then let C be maximal containing C_0. Then $c = \min\{s \in S : \psi(s) = c\}$ by Lemma 6. By Lemma 8 it suffices that c has a countable neighborhood basis.[]

10.COROLLARY. Every $S \in \underline{CL}$ is a subdirect product of $\underline{CL}$-chains (Lawson).

Proof. One has to separate two different points $s \neq t$ in S by a chain quotient. It suffices to separate st from s,resp.t. Thus one may assume $s < t$. Then consider the quotient map $x \longmapsto tx \bmod Ss : S \longrightarrow tS/sS$. We may therefore assume that

If anyone has any comments to the following question, then Stralka (and I) would like to hear it:

Suppose that $S \in CL$ and that $T \subseteq S$ is a closed chain subsemilattice. Is there a <u>canonical</u> way to find in T a subchain which is strict in T and maximal with this propoerty?

– One wishes to throw out <u>systematically</u> intervals J on which $a,b \in J$, $a < b \Rightarrow a \not\ll b$. Every such interval is contained in a maximal one. For a maximal such one has $\sup J \in J$. (If not, by maximality of J there is a $i \in J$ such that $i \ll \sup J$. By Remark 0 there is an $s \in S$ with $i \ll s \ll \sup J$. Since $\sup J \notin J$ there is a $j \in J$ with $s \leq j$. Thus $i \ll j$, a contradiction to the defining propoerty of J.) The maximal such J cover T, many may be singleton. However, there is no guarantee that they are disjoint. It is conceivable that in T one might have $a < b < c$ with $a \not\ll b \not\ll c$ but $a \ll c$. Does anyone know an example of this situation? Here is one:

If $S = I^n$ is a cube then $(x_1,\ldots x_n) \ll (y_1,\ldots,y_n)$ iff ($x_j = y_j = 0$ or $x_j < y_j$) for $j = 1,\ldots,n$, and and one easily fabricates examples of chains T with such a,b,c in 2 dimensiona. One the other hand, one knows that in this example one wishes to throw out such open intervals of a chain T which are stationary in at least one coordinate. This may not be the end of throwing out things even in this simple situation.

P.S.: Scott independently discovered the error in ATLAS which Keimel pointed out and gave the same example to illustrate it (letter 4-13-76)