SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC: Scott continuous closure operators and modal operators. More self functors to which the Scott construction applies.

A preview of a portion of the dissertation of Bill Jones.

REFERENCES: Compendium
Hofmann and Lawson, The spectral theory of distributive continuous lattices, Trans.Amer.Math.Soc.246 (1978), 285-310
Hofmann and Mislove, The lattice of kernel operators..., Math.Z.154(1977), 175-188.

In Chapter IV, Sections 3 and 4, the COMPENDIUM discusses self functors of the category INF$^\uparrow$ *and its complete subcategories, notably the categories CL and AL of continuous and algebraic lattices with Scott continuous maps preserving arbitrary infs. In particular we investigate there when such a self functor preserves projective limits, and how the Scott construction applies to it in order to create objects which are stable under the application of the functor. Several examples are given.*

This report adds some more relevant self functors to which the general theory applies. The first is the functor which associates with a continuous lattice the lattice of Scott-continuous closure operators. It is not entirely obvious in which way this assignment is functorial on CL, because the standard method, by which this functoriality is achieved in the case of Scott's function space functor does not apply here. However, by identifying our functor with the functor which associates with a continuous lattice the lattice of all Lawson closed closure systems on the lattice (in the order opposite to that of containment) we obtain functoriality. The next slight problem is that the supposedly convenient sufficient conditions given in COMPENDIUM IV-3.13 which guarantee that a self functor preserves projective limits, do not directly apply to this functor. (That, in any case, is one excuse why this functor is not now listed in the COMPENDIUM even in the exercises.) However, a sligh modification of IV-3.13 will turn out to be adequate to deal with the situation. All of this is explained in Section 1 of this Report.

The principal topic, however, of Lester W. Jones, Jr.' dissertation is the blending of continuous lattice theory with cHa theory- to "frame" continuous lattice theory, as it were. In this context the relevant category is CH, *the category of continuous Heyting algebras with CL-morphisms whose lower adjoints are Heyting algebra homomorphisms. We know from the COMPENDIUM and " The spectral theory" that CH-morphisms are characterized as those CL-morphis between continuous Heyting algebras which respect spectra. The category CH is equivalent to t category of locally quasicompact sober spaces and proper maps (in the sense explained in "The spectral theory". In the theory of cHa's, the so called modal operators play a role. In Section 2 of this report we introduce the self functor of CH which associates with a continuous Heyting algebra L the continuous Heyting algebra of all Scott continuous modal operators; a modal operator on a Heyting algebra is a closure operator preserving finite infs. The self functor on CH in question is identified with the functor M which associates with a continuous Heyting algebra L the poset M(L) of all Lawson closed subsemilattices of L whose spectrum is contained in that of L. We discuss the spectrum of M(L) to the extent we know it at this time, and we show that the map min:* $M(L) \to L$ *is an isomorphism if and only if L has a* T_1 *spectrum. Thus the modal operator functor is stationary on continuous Heyt algebras iff the spectrum is* T_1*; the abstract parallel is that the modal operator functor is stationary on a cHa iff it is Boolean.*

In Section 3 we give an abstract of an analysis of the Scott function space functor on continuous Heyting algebras, i.e. the category CH, and how the Scott projective limit construction translates into the category of locally quasicompact sober spaces. In particular, we calculate the spectrum of $\mathrm{Funct}^{\sim} L$ *for a continuous Heyting algebra L.*

1) Scott continuous closure operators- another functor to which the Scott projective limit process applies.

For ths purposes of this report we take for granted what is said in the compendium about the connection between closure operators and closure systems. We also refer to SCS Memo Mislove 12-8-76. We use the following notation:

1.1.DEFINITION. Let L be a complete lattice. Then $c(L)$ denotes the poset of all closure operators $j: L \longrightarrow L$ which are Scott continuous; the order is defined pointwise. We denote with $C(L)$ the poset of all posets $A \subseteq L$ which are closed under arbitrary infs and directed sups, and we give $C(L)$ the opposite order of that which is induced by containment. Thus if $A, B \in C(L)$, then the inf $\{A,B\}$ is just $AB = \{ab: a \in A, b \in B\}$ with $ab = \inf\{a,b\}$.

By way of summary, we record what we know and take for granted:

1.2.PROPOSITION. 1)The map $j \longmapsto j(L): c(L) \longrightarrow C(L)$ is a well-defined poset isomorphism with $A \longmapsto j_A: C(L) \longrightarrow c(L)$, $j_A(x) = \inf(\uparrow x \cap A)$ as inverse.

2) If L is continuous, then $c(L)$ and $C(L)$ are continuous.

3) If $S, T \in$ CL and $f: S \longrightarrow T$ is in CL, then for each $A \in C(S)$ we have $f(A) \in C(T)$, and if we define $C(f): C(S) \longrightarrow C(T)$ by $C(f)(A) = f(A)$, then $C(f)$ is a CL-map. Moreover, C: CL $\longrightarrow$ CL is a functor.

4) The lower adjoint of $C(f)$ given by ~~is~~ $C(f)^{*}(A) = f^{-1}A$ □

Remark. 4) shows that $C(f)$ preserves arbitrary infs; the preservation of directed sups amounts to the preservation of the intersection of a filter basis of (Lawson) compact sets under the application of a (Lawson) continuous function. This is the most convenient way to ascertain 3).

Now we have a self functor C of CL, and we adress the question whether it preserves projective limits and thereby qualifies for Scott's construction as it is outlined in Compendium IV - 4. Compendium

IV - 3 gives sufficient conditions for a self functor of $\mathrm{INF}^{\uparrow}$ and its relevant complete subcategories to preserve projective limits. We verify these conditions. Before we do this note that the isomorphic functor c is less convenient to handle: It is probably worth noticing that $c(f)(j)$ is not, as one might expect after some experience with Scott's construction, $fj\hat{f}$, but is given by $c(j)(t) = \inf(\uparrow t \cap f(j(S)))$ only if f is surjective do the two functions $c(j)$ and $fj\hat{f}$ agree,

We now inspect Compendium IV -313 and see to which degree it is applicable in the present situation.

First let $g: L \longrightarrow S$ be in <u>CL</u>. Then $C(g)^{\wedge} C(g)(A) = g^{-1}g(A)$ for all $A \in C(L)$ by 1.1.4. Similarly, if $h: T \longrightarrow L$ is in <u>CL</u> , then $C(h)C(h)^{\wedge}(B) = hh^{-1}(B) = h(T) \cap B$ for all $B \in C(L)$.

Recall that the kernel relation ker g of g is given by $(x,y) \in \ker g$ iff $g(x) = g(y)$. If this relation holds, then also $\hat{g}g(x) = \hat{g}g(y)$. Conversely, if this latter relation holds, then $g(x) = g\hat{g}g(x) = g\hat{g}g(y) = g(y)$ by Compendium 0-3.16 . Thus $\ker g = \ker \hat{g}g$. Now $g^{-1}g(A)$ is the saturation of A w.r.t. ker g; hence

$$\text{(i)} \quad C(g)^{\wedge}C(g)(A) = \ker \hat{g}g\text{-saturation of } A, \text{ for all } A \in C(L).$$

Next we claim that $h(T) = h\hat{h}(L)$: Indeed the right side is trivially in the left; however, if $x \in h(T)$, then $x = h(y)$ for some $y \in T$ and then $x = h(y) = h\hat{h}h(y) = h\hat{h}(x)$ by 0 -3.16 . This shows:

$$\text{(ii)} \quad C(h)C(h)^{\wedge}(B) = h\hat{h}(L) \cap B \text{, for all } B \in C(L).$$

Let us take stock of the situation. We want to apply IV -3.13, and we do not seem to be able to do exactly that since we have difficulty defining the map π or else to show that certain cnadidates satisfy (A1). The essence of conditions (A2) and (A3) is that $(Fg)^{\wedge}(Fg)$, and $(Fg)(Fg)^{\wedge}$ depend only on gg,respectively, gg. This we know to be true for the functor C by (i) and (ii) above. But where do we go from here?

Firstly we notice, that the function π in IV - 3.13 is never applied to anything but Scott continuous maps $L \longrightarrow L$ of the form $\hat{g}g$ or $g\hat{g}$ with $g \in \mathrm{INF}^{\uparrow}$, and these are precisely the Scott continuous kernel operators and closure operators, respectively.(See 0-3.8 ff and various places in I-2, IV-1 Exercises.) We denoted the set of Scott continuous closure operators c(L) above and clos(L) in the Compendium; in the Compendium we write ker (L) for the Scott continuous kernel operators.(See also Hofmann and Mislove :" The lattice of kernel operators...", Math.Z.154 (1977), 175-188.) We have made the following

1.3.<u>OBSERVATION</u> . Corollary IV-3.13 of the Compendium remains intact for a function $\pi: \mathrm{clos}\, L \cap \ker L \longrightarrow C^{FL}$ under otherwise unchanged conditions.□

Now we first define π on ker L . We take C(L) for the lattice C of IV-3.13, and we notice that for $f\in$ ker L , the set $f^{-1}f(A)$ = ker f -saturation of A is in C(L) for all $A\in$ C(L); reason: each f ker L is of the form $\hat{g}g$ with some $g\in$ CL (0-3.10 and I-2) and $f^{-1}f(A) = g^{-1}g(A)$. Thus we may define

$$\pi : \ker L \longrightarrow C(L)^{C(L)}, \quad \pi(f)(A) = f^{-1}f(A).$$

Then IV-3.13 (A2) is satisfied by (i) above. We now verify (A1).

Let $f = \sup f_j$ in ker L with a directed family f_j in ker L. Fix $A\in C(L)$. We must show that

$$f^{-1}f(A) = \sup f_j^{-1}f_j(A) \quad \text{in } C(L), \text{ i.e. } f^{-1}f(A) = \bigcap f_j^{-1}f_j(A).$$

We decompose each f_j into $\hat{g}_j g_j$ where $g_j : L \to f(L)$ is the corestriction of f and $\hat{g}_j : f(L) \to L$ is the inclusion; similarly we write $f = \hat{g}g$.

Now let $x \in f_j^{-1}f_j(A)$; we must show that $x \in f^{-1}f(A)$; the other inclusion is clear. Now for each j there is an $a_j \in A$ with $f_j(x) = f_j(a_j)$, i.e. with $a_j \in f_j^{-1}f_j(x)$. The filterbasis of the compact semilattices $f_j^{-1}f_j(x) \cap A$ thus has a nonempty intersection ; let $a \in A$ be an element in it . Then $f_j(x) = f_j(a)$

for all j . We pass to the sup and obtain $f(x) = f(a)$. Thus $x \in f^{-1}f(A)$ as was claimed.

Next we define π on clos(L). We define

$$\pi : \mathrm{clos}(L) \longrightarrow C(L)^{C(L)}, \quad \pi(f)(A) = f(\overset{L}{\cancel{A}}) \cap A .$$

Since $f(L) \subseteq C(L)$, this function is well defined; it satisfies IV-3.13 (A3) by(ii) above. We note that it satisfies (A1): By 1.2.1 , the function $f \longmapsto f(L)$ is an isomorphism from clos L to C(L); hence a directed set f_j with supremeum f gives a filterbasis of compact semilattices $f_j(L)$ with intersection f(L). Thus $\pi(\sup f_j)(A) = \bigcap f_j(L) \cap A = \sup \pi(f_j)(A)$.

We now have the hypotheis of IV-3.13 in the modification of 1.3 above satisfied, and therefore have proved the following result:

1.4. THEOREM. The functor C:CL $\longrightarrow$ CL preserves projective limits and the injectivity and surjectivity of morphisms.

Proof. IV-3.13 and 1.3 above. □

This completes the discussion of the functor C in the spirit of Compendium IV -3; we now turn to Scott's construction as expounded in IV -4. We first notice that there is a natural surmorphism $m_L: C(L) \longrightarrow L$:

1.5. Proposition. The function $m_L: C(L) \longrightarrow L$ given by $m_L(A) = \min A$ is a natural Cl-morphism whose lower adjoint is given by $x \longmapsto \Uparrow x$. In particular, every continuous lattice is in a functorial fashion a C-algebra (L, m_L) (see IV- 4.3).

The proof of these facts is straightforward. □

Now we can apply the Scholium IV -4.9 and obtain

1.6.THEOREM. There is a functorial retraction from the category CL_C of C-algebras over CL to the full subcategory $CL_C^{\,o}$ of C-algebras (L,p) for which $p: C(L) \longrightarrow L$ is an isomorphism. This retraction is in fact a right reflection and associates with a given natural C-algebra (L,p) a C-algebra $(\tilde{C}L, \tilde{m}\,\tilde{p})$ such that there is a quotient map $p': (\tilde{C}L, \tilde{p}) \longrightarrow (L,p)$.

There is a functorial construction whereby every continuous lattice L is a quotient of a continuous lattice $\tilde{C}L$ which is naturally isomorphic to the lattice of its Scott continuous closure operators.

If L is an algebraic lattice, then $\tilde{C}L$ is an algebraic lattice, too.

There is nothing to prove with all of the work done in IV-4.9; we should point out that the functor C maps AL into AL , and as a consequence $\tilde{C}L \in AL$ if $L \in AL$

2) Scott continuous modal operators.

2.1DEFINITION. Let L be a cHa. Then a modal operator ~~xx~~ on L is a closure operator $c: L \longrightarrow L$ preserving finite infs.

2.2.OBSERVATION. Let $c: L \longrightarrow L$ be a closure operator on a cHa.

Let

$$\begin{tikzcd} L \arrow[rr, "c"] \arrow[dr, "d"'] & & L \\ & c(L) \arrow[ur, "g"'] & \end{tikzcd}$$

be the canonical decomposition.

Then the following statements are equivalent:

(1) c is a modal operator.

(2) d preserves finite infs, i.e. d is a morphism of Heyting algebras.

These conditions imply

(3) Spec c(L) $\subseteq$ Spec L.

If Spec c(L) is order generating in c(L), then all three conditions are equivalent.

Proof. (1) <=> (2): Since the inclusion map g is an upper adjoint, c(L) is inf-closed in L. Thus c preserve finite infs iff d preserves finite infs. As a lower adjoint, the corestriction d will always preserve arbitrary sups.

(2) =>(3) (and conversely if the primes separate the ~~xx~~ points of c(L)) follows from IV-1.

Under the representation which identifies closure operators with their images in L (the so-called closure systems), the modal operators correspond to inf-closed subsets which are Heyting algebras in their own right and for which the canonical retraction preserves finite products. In the case of continuous lattices L this may be expressed in a simpler fashion: In this case we know that c(L) is a continuous lattice and so the primes separate on c(L). Thus the modal operators on a continuous Heyting algebra correspond to the closure systems which are Heyting algebras and whose spectrum is contained in the spectrum of L.

However, ~~just as in the case of arbitrary closure operators,~~ in the context of continuous distributive lattices the relevant--theory is that of Scott continuous modal operators; this is analogous to the study of arbitrary closure operators on arbitrary continuous lattices where the relevant contributions come from the assumption of Scott continuity . The theory of modal operators on cHa's is amply worked out in appropriate sources (see e.g. Simmons, " A Framework for Topology"). We concentrate here on Scott continuous modal operators on a continuous Heyting algebra for which the $<<$-relation is multiplicative; i.e., $a << x$ and $b << y$ implies $ab << xy$.

2.3. <u>DEFINITION</u>. Let L be a continuous Heyting algebra . Then $m(L)$ denotes the poset of all modal operators $j:L \to L$ which are Scott continuous; the order is defined pointwise. We denote with $M(L)$ the poset of all posets $A \subseteq L$ which are closed under arbitrary infs and directed sups (i.e., subalgebras in the sense of I-2) and which are Heyting algebras in their own right satisfying Spec $A \subseteq$ Spec L; the order on $M(L)$ is that induced from $C(L)$.

The following information is not too hard to establish:

2.4. <u>PROPOSITION</u>. 1) The map $j \mapsto j(L) : m(L) \hookrightarrow M(L)$ is a well-defined poset isomorphism with $A \mapsto j_A: M(L) \longrightarrow m(L)$ as its inverse, where j_A is defined as in 1.2.

2) If the $<<$-relation is multiplicative, then $m(L)$ and $M(L)$ are continuous Heyting algebras.

3) If S and T are in <u>CH*</u> , the category of continuous Heyting algebras and <u>CL</u>-maps preserving the spectrum, and if $f: S \to T$ is in <u>CH*</u>, then for each A in $M(S)$ we have $f(A)$ is in $M(T)$, and if we define $M(f): M(S) \to M(T)$ by $M(f)(A) = f(A)$ then $M(f)$ is a <u>CH*</u> -map. Moreover, M: <u>CH*</u> $\to$ <u>CH*</u> is a functor.

4) The lower adjoint of $M(f)$ is given by $M(f)^\wedge(A) = f^{-1}(A)$.

Proof. We discuss a portion of 2) and 3) only; the remainder is straightforward.

2) Recall from I-2.16 that there is a Scott continuous kernel operator $k : (L,L) \to (L,L)$ whose image is $[L \to L]$ and is defined by $k(f)(x) = \sup f(\Downarrow x)$ for all f in (L,L) and x in L. We show that when the $<<$ -relation is multiplicative the restriction of

k to $J(L)$, the continuous Heyting algebra of modal operators, carries $J(L)$ to $M(L)$. Therefore, $M(L)$ is a continuous lattice and a Heyting subalgebra of $J(L)$ since finite infs are the same in $J(L)$ and $M(L)$.

To see how this works, suppose j is in $J(L)$, then we must show that $k(j)$ is a closure operator since I- 2.16 makes it clear that $k(j)$ is Scott continuous:

i) For the inflationary property, $x \leq k(j)(x)$:

If y is in $\twoheaddownarrow x$, then $y \leq j(y) \leq j(x)$ so

$$x = \sup \twoheaddownarrow x = \sup \{y \mid y \in \twoheaddownarrow x\} \leq \sup \{j(y) \mid y \in \twoheaddownarrow x\} = \sup j(\twoheaddownarrow x) = k(j)($$

ii) For the idempotency, $k(j)^2 = k(j)$:

$$k(k(j))(x) = \sup(\twoheaddownarrow k(j)(x)) = k(j)(x), \quad \text{since} \quad y = \sup \twoheaddownarrow y.$$

iii) Now if the $\ll$-relation is multiplicative, we show $k(j)$ is a modal operator for a modal operator j.

We calculate

$$k(j)(xy) = \sup j(\twoheaddownarrow xy) = \sup j(\twoheaddownarrow x \twoheaddownarrow y) = \sup j(\twoheaddownarrow x) j(\twoheaddownarrow y).$$

But by 0-4.2 this sup becomes

$$\sup j(\twoheaddownarrow x) j(\twoheaddownarrow y) = \sup j(\twoheaddownarrow x) \sup j(\twoheaddownarrow y) = k(j)(x) \cdot k(j)(x).$$

3) We must show that for A in $M(S)$ we have $f(A)$ in $M(T)$. From 1.2 we know that $f(A)$ is in $C(T)$; it remains to show that Spec $f(A) \subseteq$ Spec T. Let q be in Spec $f(A)$, We consider the restriction and corestriction $g: A \to f(A)$ of f. This is a surjective <u>CL</u>-map, and there is an a in A with $g(a) = q$. Then by THE NEW LEMMA (see SCS Hofmann Watkins 5-30-79) there is a prime p in Spec A with $a \leq p$ and $g(p) \leq q$. Then $q = f(a) \leq f(p) \leq q$. Thus $g(p) = q$. But Spec $A \subseteq$ Spec S since $A \in M(S)$, and so $p \in$ Spec S. Now $q = g(p) = f(p)$. But f(Spec S) $\subseteq$ Spec T since $f \in$ <u>CH*</u>. Thus $q \in$ Spec T as was to be shown. []

It seems reasonable to expect $m(L)$ and $M(L)$ to be continuous Heyting algebras without the added restriction of $\ll$ being multiplicative but a proof is lacking at present.

2.5. LEMMA. The restriction $n_L : M(L) \longrightarrow L$ of the CL-morphism $m_L : C(L) \longrightarrow L$ (given by $n_L(A) = \min A$) is a CH-morphism.

Proof. We show that the adjoint of n_L preserves finite infs: But $(\uparrow x)(\uparrow y) = \uparrow xy$ by distributivity, which proves the assertion. □

It now follows that n_L maps Spec M(L) into Spec L. We now determine Spec M(L).

2.6. LEMMA. Let $A \in M(L)$. Then $A \in$ Spec M(L) if $A = \{a,1\}$ with $a \in$ Spec L; and if $A \in$ ~~Sepx~~ Spec M(L), then $\min A \in$ Spec L.

Proof. Since M(L) is distributive, $A \in$ Spec M(L) iff $a = \min A \neq$ and A is irreducible in M(L), iff $BC = A$ implies $B = A$ or $C = A$ for $B,C \in M(L)$. Clearly $\{a,1\}$ is irreducible. Now assume that A is irreducible. Let ~~xxxx~~ $a < a' \in A$. ~~Then there is an $x \in L$ with $x \ll a'$ and $a \notin \uparrow x$, since L is continuous. We set $B = (A \cap \uparrow x) \cup \{1\}$ and $C = A \cap \uparrow x$.~~

If $a = \min A$ were not irreducible, then $a = bc$ for some $b,c > a$, and thus $A = (A \cap \uparrow b)(A \cap \uparrow c)$ with $A \neq A \cap \uparrow b$, $A \cap \uparrow c$, and this is impossible, since $A \cap \uparrow b$, $A \cap \uparrow c \in M(L)$. Hence $a \in$ Spec L. □

At this moment we do not yet know an example of a continuous ~~xxx~~ Heyting algebra S such that 1) $0 \in$ Spec S, 2) S has more than 2 points, 3) $S \in$ IRR M(S).

Let us look at the hull-kernel topology on Spec M(L): The closed sets are precisely the hulls. Hence:

2.7. LEMMA. The closed sets in Spec M(L) are precisely the sets given via an arbitrary $B \in M(L)$ through

$$h(B) = \{ A \in \text{Spec } M(L) : A \subseteq B \}.$$

Since all $A \in$ Spec M(L) with $A = \{\min A, 1\} = \max \hat{n}_L n_L$ are maximal in Spec M(L), their singleton closures are singleton, and ~~thex~~ all closed singletons in Spec M(L) are so obtained. □

For a cHa L denote with Max L $\subseteq$ Spec L the set of all maximal elements in Spec L; this set may be empty. It consists precisely of those elements in Spec L whose hulls are singleton.

in The hull kernel topology, Max L is always a T_1 space. ~~We have noted~~ In general it will not be sober. We have noted:

2.8. Observation. $n_L \mid \operatorname{Max} M(L) : \overset{\text{Max}}{/} M(L) \longrightarrow$ Spec L is a bijective continuous map from a T_1-space onto Spec L.

(In fact, Proposition 2.5 in "The spectral Theory of Distributive Continuous Lattices, Homann-Lawson Trans.Amer.Math.Soc.246, (1978), p. 289 shows:

2.9. REMARK. For a cHa L a set $A \subseteq$ Max L is closed an irreducible iff A = ~~h(x)~~ $\uparrow x \cap$ Max L for some $x \in$ Spec L, such that $x = \inf \uparrow x \cap$ Max L.

In particular, if Max L is order generating, then $A \subseteq$ Max L is closed irreducible iff ~~iff~~ it is a hull of an element in Spec L. Consequently, in this case, Max L is sober iff Spec L = Max L, and if L is a continuous Heyting algebra, this then means that Max L is locally quasicompact sober and T_1.)

~~By the previous remark, Max M(L) is sober iff Max M(L) is Spec M(L), because Max M(L) is order generating in M(L).~~

The functor $M: CH^* \longrightarrow CH^*$ is a self functor to which ~~to which~~ the Scott process along the lines of Section 1 above applies as will be shown in Jones' Dissertation. At this point we would like to obsdrve, however, that M fixes L (up to isomorphism) iff Spec L is T_1. More precisely:

2.10. THEOREM. Let L be a continuous Heyting algebra. Then the following statements are equivalent:

(1) Spec L is T_1, i.e. Spec L = Max L, i.e. Spec L is and antichain in L.

(2) $n_L: M(L) \longrightarrow L$ is an isomorphism.

Proof. (2) =>(1). By (2) the function induced by n_L on the spectra gives an isomorphism Spec M(L) $\longrightarrow$ Spec L. By 2.8 this means Max M(L) = Spec M(L), and thus Max L = Spec ~~Spxxx~~ L. This is (1).

(1) => (2): Let $A \in$ Spec M(L), and set a = min A ($\in$ Spec L). (See 2.6

(1) implies that a is maximal in Spec L; since L is continuous and thus Spec L is order generating in L, this means that a is maximal in $L\setminus\{1\}$, i.e. is a co-atom. But then $A \subseteq \uparrow a = \{a,1\}$, and thus $A = \{a,1\}$. It follows that $n_L |$ Spec M(L): Spec M(L) $\longrightarrow$ Spec L is bijective. A closed set of Spec M(L) according to 2.7 is of the form

$$h(B) = \{\{a,1\}: a \in \text{Spec } L,\ a \in B\} = \{\{a,1\}: a \in \text{Spec } B\}.$$

Now let $b \in B$ and suppose that $b \leq q \in$ Spec L. Then, by the NEW LEMMA , there is a $p \in$ Spec B $\subseteq$ Spec L with $b \leq p \leq q$. However, by (1) we conclude $p=q$, i.e. $q \in$ Spec B. Thus

Spec $\uparrow B$ = Spec B, and thus $\uparrow B = B = \uparrow b$, where b= min B.
Hence

$$h(B) = \{\ \{a,1\}: a \in h(b)\}.$$

It follows that $n_L |$ Spec M(L): Spec M(L) $\longrightarrow$ Spec L is a homeomorphism, and thus n_L: M(L) $\longrightarrow$ L is an isomorphism by Hofmann-Lawson .

By a theorem known in framed circles (i.e, among people who look at cHa's for some reason or another), the assignment which associates with a cHa the cHa of modal operators is stationary precisely when L is a Boolean algebra. The previous theorem is the "continuous" analog of this theorem.

The dissertation of Bill Jones will have more on the matter.

3) A study of Scott's function space construction for continuous Heyting algebras: A preview.

Jones' dissertation will contain information on the functor <u>Funct</u>: <u>CL</u> $\longrightarrow$ <u>CL</u> , Funct L = [L$\longrightarrow$L](see IV -3,4) when restricted to the category <u>CH</u> . Since this category is equivalent to the category of locally quasicompact sober spaces and proper maps the question is how the limit construction translates into the category of spaces via the spec-functor. This will be completely elucidated in Jones' dissertation. As a first step the isomorphism Spec $[X, \Sigma L] \cong X \times$ Spec L given in the Compendium will be investigated for its functorial properties. This information will be specialiced to the bi-functor $(S,T) \longrightarrow [S \longrightarrow T]$, and from there to the Scott functor Funct L = $[L \longrightarrow L]$.