SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC: A new Lemma on primes
and a topological characterisation of the category DCL of continuous Heyting algebras and CL -morphisms.

REFERENCES: Compendium
Hofmann and Lawson, The spectral theory... Trans.Amer.Math.Soc. 246 (1978) 285-310.
Hofmann and Thayer, Approximately finite dimensional C* -algebra Dissertationes Math.,to appear.

This is a preview of a portion of the Dissertation of Fred Watkins.

We first prove a derivative of THE LEMMA, called THE NEW LEMMA which turns out to be quite useful here and in Bill Jones' work on continuous Heyting algebras (see SCS Memo Hofmann-Jones 5-29-79).

In the investigation of the spectral theory of C-algebras one is reduced to the study of continuous Heyting algebras and maps between them which are just CL-morphisms. We know, of course, that this is not enough for a simple translation of continuous Heyting algebra theory into the topology of locally quasicompact sober spaces along the program outlined in "The spectral theory": The morphisms which one has to consider by necessity do not preserve spectra. As long as this issue remains unresolved one cannot talk about spectra of C*-algebras in a functorial fashion.*

In this report we propose a solution to this problem. We explain in which way the spectrum of a continuous Heyting algebra can be made functorial on the big category, and what the range category has to be.

A few comments will indicate the thrust of Watkins' dissertation.

1) A new Lemma on primes

We propose a new lemma on primes which is proved with the aid of THE LEMMA. The new lemma will be applied widely in the remainder of the report and in SCS Memo Hofmann - Jones 5-29-79.

1.1. REMINDER. Let S be a continuous Heyting algebra (ctHa). Then for all $s \in S$ we have $\operatorname{Spec} \uparrow s = \uparrow s \cap \operatorname{Spec} S$.

1.2. LEMMA . Let S be a ctHa and T a complete lattice. Suppose that $f: S \longrightarrow T$ is an $\mathrm{INF}^{\uparrow}$-morphism such that $f^{-1}(0) = \{0\}$. If $0_T \in \operatorname{Spec} T$, then $0_S \in \operatorname{Spec} S$.

Proof. In S we have $0_S = \inf \operatorname{Spec} S$, whence $0_T = f(0_S) = f(\inf \operatorname{Spec} S) = \inf f(\operatorname{Spec} S)$. Thus by THE LEMMA (Compendium V-1.1), there is a $p \in (\operatorname{Spec} S)^-$ such that $f(p) = 0$. By $f^{-1}(0) = \{0\}$ we have $p = 0$, and thus $0 \in (\operatorname{Spec} S)^-$. Now $(\operatorname{Spec} S)^- \cup \{1\} = \Psi\mathrm{PRIME}\ S$ (Compendium V -3.5 ff). Thus we know that 0 is a pseudoprime in S. By definition (I-3.24) this means that there is a prime ideal I of S with $0 = \sup I$. But then $I = \{0\}$ and thus 0 is prime. □

1.3. THE NEW LEMMA. Let S be a continuous Heyting algebra and T a complete lattice. Let $f: S \longrightarrow T$ be an $\mathrm{INF}^{\uparrow}$-morphism and q a prime of T. If for some $s \in S$ we have $f(s) \leq q$, then there is a $p \in \operatorname{Spec} S$ such that $s \leq p$ and $f(p) \leq q$.

Proof. The set $\downarrow q$ is a prime ideal, thus the zero of the Rees quotient $T' = T/\downarrow q$ is prime. If we let $f': S \longrightarrow T'$ be the composition of f with the quotient map $T \longrightarrow T/\downarrow q$, then we must find a prime $p \in \uparrow s \cap \operatorname{Spec} S$ with $f'(p) =$ zero, for then $f(p) \leq q$. Thus w.l.g. we assume that $q = 0 = f(s)$.

Now $f^{-1}(0)$ is closed in S under arbitrary infs and directed sups; let m be a maximal element of $f^{-1}(0)$ with $s \leq m$ and set $S' = \uparrow m$. Then m is the zero of S' and $f(m) = 0$. If we find a $p \in \operatorname{Spec} S'$ with $f(p) = 0$ we are done, for then $p \in \operatorname{Spec} S$ by Reminder 1.1 and $s \leq m \leq p$. Moreover, $S' = \uparrow m$ is a ctHa. Thus we may assume w.l.g that $S' = S$, i.e. that $s = 0$ and that 0 is maximal in $f^{-1}(0)$. But this means $f^{-1}(0) = \{0\}$. But now Lemma 2 applies and proves the claim of the new lemma. □

The following is a first application which we will use presently.

1.4. LEMMA. Let S be a continuous Heyting algebra and R and T complete lattices with primes r and q, respectively. Suppose that $f: R \longrightarrow S$ is any map and that $g: S \longrightarrow T$ is an $\mathrm{INF}^{\uparrow}$ - morphism. If $gf(r) \leq q$, then there is a $p \in \operatorname{Spec} S$ such that $f(r) \leq p$ and $g(p) \leq q$.

Proof. We apply THE NEW LEMMA to $g: S \longrightarrow T$ with $s = f(r)$. □

2) The functor Spec

In this Section we deal with the category DCL of continuous Heyting algebras and CL -morphisms between them. This category is to be recognized as being larger than the category CH of continuous Heyting algebras and CL-maps which are upper adjoints of cHa morphisms. Our eventual goal is to identify the category DCL on the space level. We know the right objects to look for: They are the locally quasicompact sober spaces. But what are the morphisms between them? We also know the assignment on the object level,namely, the associating of the spectrum to a ctHa. In which way can this assignment be extended to a functor defined on DCL? This IS important for the applications to C*-algebras, since we do have a contravariant functor from C* to DCL given by the lattice of closed two sided ideals. If Spec is functorially defined on DCL then we have finally found the way in whic the spectrum of an arbitrary C*-algebra is functorial (modulo technicalities involving the possible difference between primitive ideals and closed prime ideals).

Our first definition is an extension of the category SET of sets and functions.

2.1. DEFINITION. Let X and Y be sets. A multivalued function (mvf) $F:X \longrightarrow Y$ is a binary relation $F \subseteq X \times Y$ with $pr_X F = X$. For $A \subseteq X$ and $B \subseteq Y$ we write $FA = F(A) = pr_Y(F \cap pr_X^{-1}A)$ and $F^{-1}B = F^{-1}(B) = pr_X(F \cap pr_Y^{-1}B)$. Finally we write $F^{[-1]}B = \{x \in X: F(x) \subseteq B\}$.
If $F:X \longrightarrow Y$ and $G:Y \longrightarrow Z$ are mvf's, then the relational product $G \circ F = \{(x,z)$: there is a y such that $(x,y) \in F$ and $(y,z) \in G\}$ is a mvf.
The mvf's form a category with sets as objects and the relational product as composition. We call it the category of sets and multivalued functions and denote it MSET. □

Now we introduce the analog of MSET for the category TOP of topological spaces and continuous maps.

2.2. DEFINITION. The mvf $F:X \longrightarrow Y$ between topological spaces X and Y is called continuous iff $F\bar{A} \subseteq (FA)^-$ for all $A \subseteq X$. It is called continuous and closed iff $F\bar{A} = (FA)^-$ for all $A \subseteq X$. It is called proper iff it is continuous and closed and the set $F^{-1}Q$ is quasicompact for each quasicompact saturated set $Q \subseteq Y$. (See V-5.2) The category MTOP has topological spaces as objects and continuous mvfs as morphisms; composition is as in MSET. The category LOC is the category of locally quasicompact sober spaces and proper mvf's, which have the additional property that they map points to closed sets. □

2.3.LEMMA. Let S and T be ctHa's and $f:S \to T$ a DCL map. The binary relation $\{(s,t) \in S \times T: f(s) \leq t\}$ is a mvf and so is its restriction and corestriction Spec f : Spec S $\longrightarrow$ Spec T. We have (Spec f)(A) = h(f(A)) for any subset $A \subseteq$ Spec S , where h denotes the hull given by $h(Y) = \uparrow Y \cap$ Spec T.
Proof straightforward. □

2.4.LEMMA. If $f:S \to T$ is a DCL -morphism, then Spec f:Spec S $\longrightarrow$ Spec T is a proper mvf.which maps points to closed sets.
Proof. i) Continuity: Let $A \subseteq$ Spec S. Let $q \in (\text{Spec } f)(\bar{A})$.This means that there is a $\bar{p} \in \bar{A}$ with $q \geq f(\bar{p})$. But $\bar{A} = h(a)$ with a = inf A by the definition of the hull-kernel topology. Thus we find $f(a) \leq f(\bar{p}) \leq q$ since f is monotone; hence $q \in h(f(A))$, and so $q \in (\text{Spec } f)(A)^-$, since $(\text{Spec } f)(A)^- = h(\inf h(f(A))$ $= h(f(A))$.

ii) Closedness: Let $q \in (\text{Spec } f)(A)^- = h(f(A)) = h(\inf f(A)) = h(f(\inf A))$ $= h(f(a))$. Thus $f(a) \leq q$. By THE NEW LEMMA 1.3 there is a $p \in \uparrow a \cap$ Spec L with $f(p) \leq q$. But $p \in h(\inf A) = \bar{A}$, and thus $q \in (\text{Spec } f)(\bar{A})$.

iii) Properness: Let $Q \subseteq$ Spec T be quasicompact and saturated Then by " The spectral theory of distributive continuous lattices" ,Prop.6.3, Q is the complement of an open filter in T. Now $p \in (\text{Spec } f)^{-1}(Q)$ means that for some $q \in Q$ we have $f(p) \leq q$, and this is equivalent to $p \in f^{-1}(\downarrow Q)$. Thus $(\text{Spec } f)^{-1}(Q) = f^{-1}(\downarrow Q) \cap$ Spec S. Since $f^{-1}(\downarrow Q)$ is the complement of an open filter in S (as $T \setminus Q$ is an open filter in T), Prop. 6.3 of "The spectral theory" applies again and shows that $(\text{Spec } f)^{-1}(Q)$ is quasicompact saturated.

iv) (Spec f)(p) = h(p) is closed. □

2.5. LEMMA. Let $f:R \to S$ and $g: S \to T$ be DCL-maps. Then
Spec gf = (Spec g) o (Spec f).
Proof. Let $(r,q) \in$ Spec gf. Then $gf(r) \leq q$. By Lemma 1.4, there is a $p \in$ Spec S such that $f(r) \leq p$ and $g(p) \leq q$,.i.e. $(r,p) \in$ Spec f and $(p,q) \in$ Spec g. Thus $(r,q) \in$ (Spec f) o (Spec g).

Now let $(r,q) \in$ (Spec g) o (Spec f); then there is a $p \in$ Spec S such that $(r,p) \in$ Spec f and $(p,q) \in$ Spec g. This means $f(r) \leq p$ and $g(p) \leq q$, and thus $gf(r) \leq g(p) \leq q$,i.e. $(r,q) \in$ Spec gf. □.

We have shown the following result:

2.6 PROPOSITION . There is a functor Spec : DCL $\longrightarrow$ LOC which assigns to a continuous Heyting algebra L its spectrum Spec L and to a DCL-map $f:S \to T$ the multivalued map Spec f: Spec S $\longrightarrow$ Spec T with Spec f = $\{(p,q) \in \text{Spec } S \times \text{Spec } T: f(p) \leq q\}$.

2.7. REMARK. The restriction of the functor Spec:DCL $\longrightarrow$ LOC to the category CH of ctHa's and upper adjoints of cHa maps which are also CL maps is the old functor Spec: CH $\longrightarrow$ LQSOB (V-5.16)

Proof. We recall that CH maps throw spectra into spectra. In addition one observ that we have a natural bijection between continuous functions $f:X \longrightarrow Y$ between sober spaces and those continuous relations $F:X \longrightarrow Y$ which map irreducible closed sets to irreducible closed sets in such a fashion that $F\{x\}^- = \{f(x)\}^-$.

2.8. PROPOSITION. Let $f:S \longrightarrow T$ be a DCL -morphism. Consider closed sets $A \subseteq$ Spec S and $B \subseteq$ Spec T and set $a = \inf A$ in S and $b = \inf B$ in T. Then the following two statements are equivalent:

(1) $f(a) = b$. (2) (Spec f)(A) = B.

Also, the following two statements are equivalent:

(I) $a = \hat{f}(b)$. (II) $A = (\text{Spec } f)^{[-1]}(B)$.

Proof. (1) $\Leftrightarrow$ $f(\inf A) = \inf B$ $\Leftrightarrow$ $\inf f(A) = \inf B$ (since f preserves infs) $\Leftrightarrow$ $\inf hf(A) = \inf B$ (since primes order generate) $\Leftrightarrow$ $hf(A) = B$ (since both hf(A) and B are closed) $\Leftrightarrow$ (Spec f)(A) = B $\Leftrightarrow$ (2).

(I) $\Rightarrow$ (II): (I) implies $b \leq f(a)$ by the definition of the lower adjoint. Thus $\inf B \leq f(\inf A) = \inf f(A)$. Thus $f(A) \subseteq h\inf B = B$ (since B is closed). This means (Spec f)(A) $\subseteq$ B, i.e. $A \subseteq (\text{Spec } f)^{[-1]}(B)$. In order to prove the reverse containment we take a p in $(\text{Spec } f)^{-1}(B)$. Then $h(f(p)) \subseteq B$, and thus $\inf B \leq \inf h(f(p)) = f(p)$. Thus $p \in f^{-1}(\uparrow \inf B)$, hence $p \geq \inf f^{-1}(\uparrow \inf B) = \hat{f}(b) = a = \inf A$, whence $p \in h(\inf A) = A$, since A is closed.

(II) $\Rightarrow$ (I) : We have $(\text{Spec } f)^{[-1]}(B) = f^{-1}(\uparrow \inf B)$ by the reasoning in the previous argument and by its converse. Thus (II) implies $a = \inf A = \inf f^{-1}(\uparrow \inf$ $= \hat{f}(b)$. □

3. The functor Γ^{op}.

In this section we find the inverse for the functor Spec.

3.1. DEFINITION. For any topological space X we write $\Gamma^{op}(X)$ for the cHa of all closed subsets of X IN THE OPPOSITE ORDER (of that given by containment) Thus $\Gamma^{op}(X) = O(X)$. We recallthat, in particular, $\Gamma^{op}(X)$ is a continuous Heyting algebra if X is an LOC -object.

Let $F:X \longrightarrow Y$ be a morphism in LOC. We define a function (!)

$$\Gamma^{op}(F): \Gamma^{op}(X) \longrightarrow \Gamma^{op}(Y) \quad \text{by} \quad \Gamma^{op}(F)(A) = F(A). \quad \square$$

3.2.LEMMA. If $F \in$ LOC, then $\Gamma^{op}(F)$ is a DCL -map.

Proof. i) $A \mapsto F(A)$ preserves infs: From $F\bar{B} = (FB)^-$ we derive $F((\bigcup A_j)^-) = (\bigcup FA_j)^-$ for any family A_j.

ii) $A \mapsto F(A)$ preserves directed sups: Let $\{A_j\}$ be a filterbasis of closed sets in X with non-empty intersection A. Then always $FA \subseteq \bigcap FA_j$. Conversely, if $x \in \bigcap FA_j$, then $F^{-1}x \cap A_j \neq \emptyset$ for all j. But $F^{-1}x^*$ is quasicompact where x^* is the quasicompact saturation of $\{x\}$. Then intersection $F^{-1}x^* \cap \bigcap A_j$ contains an element y, for which there is an $x' \in x^*$ with $(y,x') \in F$. Then $x' \in Fy \subseteq F(\bigcap A_j)$. Then $x \in \{x'\}^- \subseteq F(\bigcap A_j)$, since F is closed.
The case that $\bigcap A_j = \emptyset$ is clear. $\square$

We will now show that Γ^{op} inverts Spec up to within natural isomorphism.

3.3. LEMMA. Let $f: S \longrightarrow T$ be in DCL. Then the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ {\scriptstyle h_S}\downarrow & & \downarrow{\scriptstyle h_T} \\ \Gamma^{op}(\text{Spec } S) & \xrightarrow[\Gamma^{op}\,\text{Spec } f]{} & \Gamma^{op}(\text{Spec } T) \end{array}$$

Proof. Γ^{op} Spec f h(s) = (Spec f)(h(s)) = h(f(h(s)). Now inf h(f(h(s))= inf f(h(s) = f(inf h(s))= f(s). Also inf h(f(s)) = f(s). $\square$

3.4.LEMMA. Let $F: X \longrightarrow Y$ be an LOC morphism. Then the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad F \quad} & Y \\ {\scriptstyle \phi_X}\downarrow & & \downarrow{\scriptstyle \phi_Y} \\ \text{Spec}\,\Gamma^{op}(X) & \xrightarrow[\text{Spec}\,\Gamma^{op}\,F]{} & \text{Spec}\,\Gamma^{op}(Y) \end{array},$$

where $(A,B) \in \phi_X$ iff $\{a\}^- \in B$ for all $a \in A$.

Proof. $\phi_Y FA = \{\{y\}^- \mid y \in FA\}$. Now Spec Γ^{op} F $\phi_X A$ =
[Spec $\Gamma^{op}(F)$] $\{\{a\}^- \mid a \in A\} = \{\{y\}^- \mid \{y\}^- \subseteq F\{a\}^-$ for some $a \in A\}$.
But $y \in FA$ iff $y \in F\{a\}$ for some $a \in A$ iff $\{y\}^- \subseteq F\{a\}^-$ for some $a \in A$ since $F\{a\} \subseteq (Fa)^-$ (as images of points are closed) $= F\{a\}^-$. $\square$

3.5. THE MAIN THEOREM. The category DCL is equivalent to the category LOC under the pair of mutually inverse functors Spec: DCL $\longrightarrow$ LOC and Γ^{op}: LOC $\longrightarrow$ DCL.

Proof. Since we know from Compendium V that $h_S: S \longrightarrow \Gamma^{op}(\mathrm{Spec}\ S)$ and $\Phi_X: X \longrightarrow \mathrm{Spec}\ \Gamma^{op} X$ are isomorphisms in their respective categories DCL and LOC. Lemmas 3.3 and 3.4 show that they are natural. □

4. A preview of the dissertation of Fred Watkins.

What has been developed here is a tool to treat the spectral theory of C*-algebras in a functorial fashion. The functor which associates with a C*-algebra A the continuous Heyting algebra $\overline{\mathrm{Id}}\ A$ of closed two sided ideals and to a C*-map $f: A \longrightarrow B$ the assignment $I \longmapsto f^{-1}I: \overline{\mathrm{Id}}\ B \longrightarrow \overline{\mathrm{Id}}\ A$ is a contravariant functor from C* to DCL (See Compendium I-1.20 ff.) It will be shown that this functor transforms injective limits to projective limits; a preliminary proof of this fact is contained in Hofmann -Thayer :"Approximately finite dimensional C*-algebras". One is interested in building complicated C*-algebras from simpler ones via injective limits; AFC*-algebras are a simple case in point which is not trivial. One wishes to determine the spectrum of the limit in terms of the spectra of the approximating algebras. Through the present theory this is now translated into the problem of describing the spectrum of a projective limit in DCL if information on the approximating ctHa's is given. Watkins' dissertation will carry this program through. This generalizes the earlier attempts by Bratteli Elliott, Hofmann and Thayer and puts these precursors in the "correct" lattice theoretical framework.

There is the open question whether there is a purely lattice theoretical description of the primitivity of an ideal in $\overline{\mathrm{Id}}$ A. The dissertation will make an attempt to use the DCL -theory together with C*-algebra constructions to build an example of a non-separable C*-algebra in which there is a closed prime ideal which is not primitive.