SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

| | | | | |
|---|---|---|---|---|
| NAME: Karl H. Hofmann | Date | M | D | Y |
| | | 11 | 27 | 79 |

TOPIC: Completely distributive algebraic lattices

REFERENCES: A Compendium of Continuous lattices (and the literature quoted therein)

Further references: Areski Nait -Abdallah: Faisceaux et semantique des programmes, These d'etat 198*

---

The Compendium touches upon completely distributive lattices in various places. As people get more and more interested in continuous posets, completely distributive lattices will attract more attention in view of the close relation between the two.

The Compendium says almost nothing on completely distributive algebraic lattices; perhaps the authors of the compendium considered them too special to merit special attention. But in the same vein, they correspond bijectively to the algebraic posets via their spectral and co-spectral theory. These have been studied in the context of programming, notably by Plotkin. Completely distributive algebraic lattices also appear to play a role in the domains of algorithms of Nolin. Of course, there is a literature on these lattices, but it seems anyhow reasonable to revisit them in the light of continuous lattice theory. I offer some remarks in the following (and possibly in a subsequent) memo.

My original motivation stems from my desperate efforts to understand Nolin's domains of algorithms as axiomatized in the thèse of Nait-Abdallah, but I have not succeeded with that. However, I think that before one can settle that issue in a way satisfactory to the SCS seminar, one would have to cover some of the theory I want to discuss.

Let me remark that I have a terminological difficulty. A goodshort name is wanted for completely distributive algebraic lattices. This name is too long. The logicians have called them Kripke models, we have also called them distributive bi-algebraic lattices. In the meantime I callthem baHa's (bi-algebraic Heyting algebras).

1. Complete irreducibles revisited.

In view of what I want to say later, I propose to take a second look at complete irreducibles which are intrduced on p. 92 (of the"Compendium"- all references which are not specified are made to the "Compendium"). In an earlier version we also talked about "complete primes",but then we gave up on them, seemingly because we had no real need for them.

I think that the need will arise in the near future. A lot of thinking will go into more spectral theory and notably into the study of continuous posets. We have a pretty good idea that studying continuous posets means studying completely distributive lattices and vice versa-thanks to JD.Lawson's theorem (p.241,p.265) and to Hyphen-Hoffmann's advocating continuous posets in general topology. I have the impression that in the Compendium completely distributive lattices are generally treated as an rather narrow special case of continuous lattices and as a matter of history, by and large. It would not surprise me if the connection between continuous posets and completely distributive lattices would lead to a renaissance of completely distributive lattices. For the moment, the precise correspondence between completely distributive lattices and continuous posets in its full functorial aspects is still a project for the future; Jaime Niño is likely to have something to say about that in his dissertation.

If we look at completely distributive algebraic lattices we notice that they are not even mentioned in the compendium (or are they?). The compendium apparently treats them even more as a curiosity than completely distributive lattices themselves. Once again, the literature has much information on these, but nothing of substance appears to be on record on their relation to continuous poset theory. They relate,of course, to algebraic posets. These have been looked at by Plotkin in the context of certain programming situations; this topic does not seem to be cleaned out either. The domains of algorithms by Nolin are based on completely distributive algebraic lattices, and Batbedat's studies on monogenetic spaces have led him up against completely distributive algebraic lattices,too. I therefore think that a few things here and there should be addressed by SCS when it comes to completely distributive lattices,respectively, c.d.algebraic lattices. I want to make a few observations which pertain to the latter and which make reference to the forthcoming Thèse d' État of Areski Nait-Abdallah.

1.1 LEMMA. Let L be a complete lattice and $p \in L$. Then the following statements are equivalent:

(1) There is a (unique) element $p^+ > p$ such that $\uparrow p = \{p\} \cup \uparrow p^+$.

(2) $p < 1$ and for each subset $X \subseteq L$ the relation $p = \inf X$ implies $p \in X$.

Also the following statements are equivalent:

(I) There is a (unique) element $p^* \nleq p$ such that $L = \downarrow p \cup \uparrow p^*$.

(II) $p < 1$ and for each subset $X \subseteq L$ the relation $\inf X \leq p$ implies $p \in \uparrow X$.

(III) $p \in K(L^{op}) \cap \mathrm{Spec}\, L$.

Remark. The union occurring in (I) is clearly disjoint as a consequence of $p^* \notin \downarrow p$ .

Proof. (1) <=>(2) is used widely in I-4 and (I)<=>(II) is just as easy to see.

(I) and (II) => (III): $p \in \mathrm{Spec}\, L$ is clear from (II). Condition (I) tells us that $\uparrow_{L^{op}} p = \downarrow p$ is $\sigma(L^{op})$-open in $L^{op}$, whence $p \in K(L^{op})$. (See p .85, I-4.2.)

(III) =>(I): From $p \in \mathrm{Spec}\, L$ we know that $L \setminus \downarrow p$ is a filter, and from $p \in K(L^{op})$ we derive that $L \setminus \downarrow p = L \setminus \uparrow_{L^{op}} p$ is $\sigma(L^{op})$ -closed in $L^{op}$, and hence, after the preceding, is a Scott closed ideal in $L^{op}$. Hence it has a maximal element $p^*$ in $L^{op}$. This yields (I). □

1.2. DEFINITION. a) An element $p \in L$ satisfying (1) and (2) in 1.1 is called <u>completely irreducible</u> (p.92, I- 4.19) and the set of all complete irreducibles is called Irr L.

b) An element $p \in L$ satisfying (I), (II) and (III) is called <u>completely prime</u>. The set $K(L^{op}) \cap \mathrm{Spec}\, L$ of all complete primes will be abbreviated $\Theta^*(L)$, and the set $K(L) \cap \mathrm{Spec}\, L^{op}$ of all <u>complete coprimes</u> will be abbreviated $\Theta_*(L)$. □

1.3. Notation. If $p \in \Theta_*(L)$ then there is a unique element $\Theta^*(L)$ which we will again call $p^*$ such that L is the dijoint union of $\downarrow p^*$ and $\uparrow p$. □

1.4 PROPOSITION. The functions $p \longmapsto p^{*}: \Theta^*(L) \cup \Theta_*(L) \to \Theta^*(L) \cup \Theta_*(L)$ is an involution mapping $\Theta^*(L)$ bijectively onto $\Theta_*(L)$ (and vice versa).

Proof. Clear. $\square$

Thus complete primes and complete coprimes appear together, or not at all. Nait Abdallah calls complete coprimes "<u>éléments</u> <u>atomiques</u>" They combine the properties of being compact ~~with~~ and being ~~p~~ coprime. When are complete irreducibles completely primes ?

1.5. PROPOSITION. Let L be a complete lattice. Then $\Theta^*(L) \subseteq \mathrm{Irr}\, L$ and $p^+ = p \vee p^{*}$. If $L^{op}$ is a cHa (i.e. if L is join-continuous and distributive(p.31,0-4.3)) then $\Theta^*(L) = \mathrm{Irr}\, L$, and $p^{\circ} = (p^+ \Rightarrow_{L^{op}} p)$.

Proof. Firstly, if $p \in \Theta^*(L)$, the element $p^+ = p \vee p^{\circ}$ satisfies the requirements of 1.1(1).

Secondly, suppose that $L^{op}$ is a cHa. Then the element $p^{*} = (p^+ \Rightarrow_{L^{op}} p)$ is exactly $\max_{L^{op}} \{x \mid x \wedge_{L^{op}} p \leq_{L^{op}} p^+\}$ $=$ $\min \{x \mid x \vee p \geq p^+\}$ and this is clearly $\min\{x \mid x \nleq p\}$ .

(One may of course use join-continuity and distributivity to derive 1.1(II) from 1.1(2), but this would still leave you with the task of determining $p^{*}$ as an equationally given function of p and $p^+$ (and thus of p).)

In order to have complete symmetry such as is indicated by 1.4 the right class of lattices for the $\Theta^*$ -$\Theta_*$ theory is that of all L which are cHa's such that $L^{op}$ is also a cHa, in other words the class of meet and join continuous distributive lattices. In this class we have $\mathrm{Irr}\, L = \Theta^*(L)$ and $\mathrm{Irr}(L^{op}) = \Theta_*(L)$. This brings us near completely distributive lattices, but not quite. We have continuous lattices which are join continuous but which are not completely distributive (see pp. 316 ff. ,pp329 ff.).

Recall that a set $X \subseteq L$ is order generating iff $x = \inf(\uparrow x \cap X)$ for all x (p.70,3.8). The following must be on record somewhere, but I do not know where.

1.6. THEOREM. Let L be a complete lattice. Then the following are equivalent:

(1) $\Theta^*(L)$ is order generating.

(2) $\Theta_*(L)$ is order generating in $L^{op}$ (every element is the sup

of complete coprimes. )

(3) L is a completely distributive algebraic lattice.

Remark. For condition (3) we have numerous equivalent statements which parallell p.72, I-3.15, pp.204ff., pp.317 ff. plus all those statements which are in the literature, e.g. the following:

(4) The SUP $\cap$ INF morphisms $L \longrightarrow 2$ separate the points.

Proof. (3)$\Rightarrow$(1). By p.93, Irr L is order generating; by p.72, I-3.15, the hypotheses of 1.5 are satisfied, and so $\overset{K}{\mathrm{Irr}}\ L = \Theta^*(L)$.

Next we note that (1) is equivalent to the following

(*) For any pair of elements in L with $x^* \not\leq x$ there is a $p \in \Theta^*(L)$ with $x \leq p$ and $x^* \not\leq p$.

Evidently, this condition is equivalent to

($_*$) For any pair of elements in L with $x^* \not\leq x$ there is a $p^* \in \Theta_*(L)$ with $p^* \leq x^*$ and $p^* \not\leq x$.

But this condition is equivalent to (2). Thus (1) and (2) are equivalent.

(1) $\Rightarrow$ (3): Quick proof: Buy that (3)$\Leftrightarrow$(4), and note that (1)$\Leftrightarrow$(4) is immediate.

Proof within the Compendium: From (2) we know that $x = \sup({\downarrow}x \cap \Theta_*(L)) \leq \sup({\downarrow}x \cap K(L)) \leq x$ by 1.2b. So L is algebraic, hence continuous. By I- 3.15 we know that L is completely distributive, since $\Theta_*(L)$, hence the set of coprimes is order-cogenerating. □ *Note: We have in fact produced a proof of the equivalence of (4) with (3).*

1.10. DEFINITION. The lattices characterized in Theorem 1.9 will be called <u>bi-algebraic</u> <u>lattices</u> or <u>bi-algebraic</u> <u>Heyting</u> <u>algebras</u> (baHa). □ *Some information on these was given in HMS DUALITY (LNM 396).*

1.11. PROPOSITION. Let L be a baHa. Then Spec $L^{op}$ is an algebraic poset in the induced order with $K(\mathrm{Spec}\ L^{op}) = \Theta_*(L)$. Dually, (Spec L, $\geq$ ) is an algebraic poset with $K((\mathrm{Spec}\ L, \geq)) = \Theta^*(L)$.

Proof. By Lawson duality (p.241) we indicate only the first part of the proof. By p.241 we need only show $K(\mathrm{Spec}\ L^{op}) = \Theta_*(L)$. From 1.2b the containment $\supseteq$ is clear. Let $k \in K(\mathrm{Spec}\ L^{op})$. Then ${\uparrow}_{\mathrm{Spec}\ L^{op}}\, k$ is an open filter U in Spec $L^{op}$. Then ${\uparrow}_L k = {\uparrow}U$ is an open filter in L by p.241, V-1.11. Thus $k \in K(L) \cap \mathrm{Spec}\ L^{op} = \Theta_*(L)$.

1.12.THEOREM. Let L be a complete lattice and define $g\colon L \longrightarrow 2^{\Theta_*(L)}$ by $g(x) = \downarrow x \cap \Theta_*(L)$ and $d\colon 2^{\Theta_*(L)} \longrightarrow L$ by $d(P) = \sup P$. Then we have the following conclusions:

i) $(g,d)$ is a Galois connection.
ii) $g$ is a SUP$\cap$INF -map.
iii) The image of $g$ is the complete sublattice T of all lower sets of $\Theta_*(L)$.
iv) The image of $d$ is the set $\{x \in L \mid x = \sup(\downarrow x \cap \Theta_*(L))\}$.
v) $g$ is injective iff $d$ is surjective iff $L$ is a baHa; in this case $g\colon L \longrightarrow T$ is an isomorphism with $d|T$ as inverse.

Proof. i) $d(P) \leq x$ means $\sup P \leq x$ and this is equivalent to $P \subseteq \downarrow x \cap \Theta_*(L) = g(x)$.

ii) By i) we know that $g$ is an INF-map. Now let $X \subseteq L$. Let $p \in \Theta^*(L)$. Then $p \in g(\sup X)$ iff $p \leq \sup X$ iff $p \in \downarrow X$ (by 1.1(II)) iff $p \in \bigcup\{\downarrow x \cap \Theta^*(L)\colon x \in X\} = \sup g(X)$.

iii) If $P$ is a lower set in $\Theta_*(L)$, then $gd(P) = \downarrow \sup P \cap \Theta_*(L)$; we just saw that a $p \in \Theta_*(L)$ is in $\downarrow \sup P$ iff $p \in \downarrow P$, but $\downarrow P \cap \Theta_*(L) = P$. So $gd(P) = P$.

iv) Clear.

v) Clear from 1.6(1),iv above and p.21 ,0-3.7. $\square$

Of course, the lower sets on $\Theta_*(L)$ are the open sets of an A-discrete topology. These are the Kripke models. Conversely, every Kripke model is a baHa. One will notice that in our tables on pp.268 and 269 (this is where they will be in the book!) the Kripke models appear opposite completely distributive lattices in which $\mathrm{Spec}\, L^{op} = \Theta_*(L)$: In these tables we have a different correspondence between cHa's and spaces, namely, the one given by Spec and O. For the Kripke models, the one in 1.12 is simpler. I leave it to the next man to elaborate on all of this. Of course there are connections to several papers by Hyphen-Hoffmann, notably [1979c].

## 2. The "normal" morphisms of Nolin/Nait.

2.1.DEFINITION.(Nolin,Nait). A function $f:S \longrightarrow T$ between two baHa's is called normal iff

$$f(x) = \sup\, f({\downarrow}x \cap \Theta_*(\overset{S}{m})) \quad \text{for all } x \in S. \ \square$$

By p.112 ,II-112(5) we know that every normal function is Scott continuous. In fact we will observe more:

2.2.PROPOSITION. Let S,T be baHa's (Kripke models). Then a function $f:S \longrightarrow T$ is normal iff $f \in SUP(S,T)$.
Proof. i) Suppose that f is normal and let $X \subseteq L$. Set $\overset{S}{m} = \sup X$ and defined g as in 1.12. Then we have $f(\sup X) = f(\overset{S}{m}) =$
$\sup f({\downarrow}\overset{S}{m} \cap \Theta_*(L))$ (by 2.1) $= \sup fg(\overset{S}{m})$ (by def.of g)
$= \sup fg(\sup X) = \sup f(\bigcup g(X))$ (by 1.12) $= \sup \bigcup fg(X) =$
$\sup_{x \in X} \sup fg(x) = \sup_{x \in X} \sup f({\downarrow}x \cap \Theta_*(\overset{S}{m})) = \sup_{x \in X} f(x)$
(by 1.6(1)) $= \sup f(X)$.

ii) Suppose that $f \in SUP(S,T)$. Then $f(x) = f(\sup({\downarrow}x \cap \Theta^*(\overset{S}{m})))$ (by 1.6(1)) $= \sup f({\downarrow}x \cap \Theta_*(\overset{S}{m}))$ since f preserves sups. So f is normal by 2.1. $\square$

2.3.COROLLARY/. $f:S \longrightarrow T$ is normal iff it has an upper adjoint $\hat{f}:T \longrightarrow S$. The upper adjoint is co-normal,i.e. $\hat{f}(\overset{y}{m}) = \inf \hat{f}({\uparrow}x \cap \Theta^*(T))$.
Proof. The first assertion is a consequence of 2.2 and SUP-INF-DUALITY (p.179, 1.3). Then second assertion is just the dual of 2.2. $\square$

2.4.PROPOSITION. Let S,T be baHa's and $d:S \longrightarrow T$ a lower adjoint of $g:T \longrightarrow S$. Then the following are equivalent:

(1) $d(\Theta_*(S)) \subseteq \Theta_*(T)$.

(2) g is normal.

(3) g is a complete lattice map ($g \in SUP \cap INF$)

Furthermore,the following are equivalent:

(I) $g(\Theta^*(T)) \subseteq \Theta^*(S)$.

(II) d is a complete lattice map.

Proof. We know (2)<=>(3) by 2.2. The proof of (I)<=>(II) is dual to that of (1)<=>(2).

(1) $\Rightarrow$ (3) (i.e. $g$ preserves sups). Let $Y \subseteq T$. We always have $\sup g(Y) \leq g(\sup Y)$. Assume that $<$ holds. Then there would be a $p \in \Theta^*(L)$ such that $\sup g(Y) \leq p$ and $p^* \leq g(\sup Y)$. Then second inequality means $d(p^*) \leq \sup Y$. By (1) and 1.1(II) there is a $y \in Y$ such that $d(p^*) \leq y$, i.e. $p^* \leq g(y)$. Then $p^* \leq \sup g(Y)$, and that contradicts $\sup g(Y) \leq p$.

(3) $\Rightarrow$ (1). Let $q \in \Theta^*(T)$ and $\inf X \leq g(q)$. Then $d(\inf X) \leq q$. But $d(\inf X) = \inf d(X)$ by (3), and so $d(x) \leq q$ for some $x \in X$. This means $x \leq g(q)$. This shows $g(q) \in \Theta^*(S)$. □

Jaime Niño will develop a duality theory between algebraic posets and baHa's with complete lattice maps as morphisms based on this set-up.

~~2.5. LEMMA. Let X be a sober space such that O(X) is a baHa (i.e. X is an algebraic poset such that O(X) is its Scott topology). Let L be completely distributive. Then [X, L] is completely distributive.~~
~~Proof. Let U be a complete co-prime in O(X) and p a co-prime in L.~~

2.5. LEMMA. i) Let $X$ be a topological space obtained from a continuous poset by taking its Scott topology and let $L$ be a completely distributive lattice. Then $[X, \Sigma L]$ is a completely distributive lattice. If $X$ is obtained from an algebraic poset and $L$ is a baHa, then $[X, \Sigma L]$ is a baHa.
Remark. These conditions are also necessary.
Proof. We invoke p.264, V- 5.20 and p.241, V-1.10 and p.265, V-5.23. From V-5.20 we know that $\mathrm{Spec}\,[X, \Sigma L] = X \times \mathrm{Spec}\, L$. In the specialisation order $\Omega(X \times \mathrm{Spec}\, L)$ is a continuous poset if $\Omega X$ is a continuous poset and $L$ is a continuous lattice. Then $[X, \Sigma L]$ is a completely distributive lattice by V-5.23 in view of the general spectral theory of continuous lattices. The second part of the Lemma is proved analogously. □

(Does anyone know an elementary proof for this Lemma?)

2.6. COROLLARY. If $S$ and $T$ are baHa's, then $[S \longrightarrow T]$ is a baHa. In particular, the functor Funct of p.218, IV- 3.18 preserves baHa's.
Remark. In the same vein, completely distributive lattices are preserved. □

2.7. COROLLARY. Let L be a baHa. Then so is $\mathrm{Funct}^{\sim} L$ (p.232, IV-4.12).
Remark. The analogous statement holds for completely distributive lattices.
Proof. Completely distributive lattices form a complete category relative to INF $\cap$ SUP maps on account of the equational definition of completely distributive lattices (p.59, I-2.4). Since AL is a complete category, we conclude that baHa's form a complete category relative to INF$\cap$SUP maps. By p.231, IV- 4.11, the fixed point construction giving Funct L does not lead outside this category. Hence $\mathrm{Funct}^{\sim} L$ is a baHa.$\square$

2.8. LEMMA. Let S,T be baHa's. Define $k:[S\to T]\to[S\to T]$ by $(kf)(x) = \sup f(\downarrow x \cap \Theta_*(S))$. Then k is a Scott-continuous kernel operator, whose image is SUP(S,T), the set of normal maps $S \longrightarrow T$.
Proof: Routine.$\square$

This shows, that SUP(S,T) is a continuous lattice. We would like to show that it is a baHa.

2.9. LEMMA. Let S,T be baHa's. The function
$$(p,q) \longmapsto \chi_{S\setminus\downarrow p} \vee \mathrm{const}_q : \Theta_*(S)\times\Theta^*(T) \longrightarrow \Theta^*(\mathrm{SUP}(S,T))$$
is a well defined bijection, *and $\Theta^*(SUP(S,T))$ is order generating.*
Proof. We write $p\#q = \chi_{S\setminus\downarrow p} \vee \mathrm{const}_q$; i.e. $(p\#q)(x) = q$ if $x \leq q$ and $= 1$ otherwise.

Firstly, we show that $p\#q$ is well-defined, i.e. that $p\#q$ is completely prime in SUP(S,T). Suppose that $f_j$ is any family in SUP(S,T) and that $\inf f_j \leq p\#q$.
Case i. $x \leq p$: then $\inf f_j(x) \leq (p\#q)(x) = q$, and so there is some $j(x)$ such that $f_{j(x)}(x) \leq q$.
Case ii. $x \nleq p$: then $(p\#q)(x) = 1$, and thus $f_j(x) \leq (p\#q)(x)$ for all j.
Now we have $f_{j(p)}(p) \leq q$. By monotonicity we have $x \leq p \Rightarrow f_{j(p)}(x) \leq f_{j(p)}(p) \leq q = (p\#q)(x)$ and also $x \nleq p \Rightarrow f_{j(p)}(x) \leq (p\#q)(x)$ by Case ii. Thus $f_{j(p)} \leq p\#q$. This shows that

Next we show that $\Theta_*(S) \# \Theta^*(T)$ is order -generating in SUP(S,T).
Let $f:S \longrightarrow T$ be normal. We note that $(\hat{f}(q)\#q)(x) = q$ if $x \le \hat{f}(q)$ iff $f(x) \le q$ , and $= 1$ if $f(x) \not\le q$. Thus $\hat{f}(q)\#q \ge f$ for all q.
Now we set $F = \inf\{\hat{f}(q)\#q : q \in \Theta^*(T)\}$. Then $f \le F$. Suppose that there were an x with $f(x) < F(x)$. Then there would be a $q \in \Theta^*(T)$ such that $f(x) \le q$ , but $F(x) \not\le q$. But $f(x) \le q$ implies $F(x) \le (f(q)\#q)(x) = q$, a contradiction. Thus $f = \inf\{f(q)\# q: q \in \Theta^*(T)\}$.
Finally, if $s \in S$ is arbitrary, then $s = \sup(\downarrow s \cap \Theta_*(S))$.
Thus $s\#q = \inf\{p\#q \mid p \le s,\ p \in \Theta_*(S)\}$: Indeed if $x \in S$, then $x \le s$ implies $(s\#q)(x) = q$ on one hand and $q \le \inf\{(p\#q)(x) \mid s \ge p \in \Theta_*(S)\}$ $\le \inf\{p\#q(x) \mid x \ge p \in \Theta_*(S)\} = q$ ; however, if $x \not\le s$ , then $(s\#q)(x) = 1$ on one hand and $\inf\{(p\#q)(x) \mid s \ge p \in \Theta_*(S)\} = 1$ on the other, since $s \ge p$ and $x \not\le s$ implies $x \not\le p$, and so $(p\#q)(x)=1$.
Thus f is the inf of elements p#q with $p \in \Theta_*(S)$ and $q \in \Theta^*(T)$.

Now $\Theta^*SUP(S,T) \subseteq Irr(SUP(S,T) \subseteq \Theta_*(S) \# \Theta^*(T)$ by p.92, I- 4.20.
Thus the function # is surjective; it is clearly injective. ~~q.e.d.~~ □

2.10.THEOREM. Let S and T be baHa's. Then SUP(S,T), the lattice of nomal maps from S to T is a baHa,and $\Theta^*(SUP(S,T))$ is isomorphic to $\Theta_*(S) \times \Theta^*(T)$.
Proof. This follows from 1.6 and 2.9. □
EXERCISE. Verify that the isomorphism of 2.9 and 2.10 respects the algebraic poset- and thus the topological structure.
(Hint.: Show that # is decreasing in the first argument, increasing in the second relative to the induced order structures; in the second argument and the range, the induced order is opposite to the algebraic poset (= specialisation) order. Then use p.265, V-5.23. □

2.11COROLLARY. If S and T are baHa's , then so is $S \otimes T$.
Proof. 2.10 and p.192,IV-1.44. □

Turn to p.218, IV-3.18 in the case that L is a baHa.If $g:S \longrightarrow T$ is a complete lattice morphism, then $Funct(g)(\varphi)=g\varphi\hat{g}$ preserves sups for $\varphi \in SUP(S,S)$, since g preserves sups as a complete lattice morphism. Thus the functor

$\mathrm{Funct}:\mathrm{INF}^{\uparrow} \longrightarrow \mathrm{INF}^{\uparrow}$ induces a self functor $S \longmapsto \mathrm{SUP}(S,S) \subseteq \mathrm{Funct}\,S$ on the category of baHa's with complete lattice morphisms. Call this functor $\mathrm{Funct}_b$ . The Scholium IV-4.9 on p.230 applies because of IV-4.11 on p.231. We therefore have the following

2.12. PROPOSITION. If L is a baHa, then so is $\mathrm{Funct}_b^{\sim} L$. Moreover, $\mathrm{Funct}_b$ L is naturally isomorphic to the space of its normal self-functions. There is a commutative diagram

$$\begin{tikzcd}
{[\mathrm{Funct}^{\sim} L \longrightarrow \mathrm{Funct}^{\sim} L]} \arrow[r, "\tilde{p}_L"] & \mathrm{Funct}^{\sim} L \\
\mathrm{SUP}(\mathrm{Funct}^{\sim} L, \mathrm{Funct}^{\sim} L) \arrow[u, phantom, "\cup"] & \\
\mathrm{SUP}(\mathrm{Funct}_b^{\sim} L, \mathrm{Funct}_b^{\sim} L) \arrow[u, hook] \arrow[r] & \mathrm{Funct}_b^{\sim} L \arrow[uu, hook]
\end{tikzcd}$$
$\square$

Further elucidation of this situation is to be expected in Jaime Niño's dissertation.

I conclude this section with some further remarks on the function # which we have used in the proof 2.9. But first we define this function on a larger domain:

2.13. DEFINITION. Let S,T be ~~baHa's~~ complete lattices. Define $\#: S \times T \longrightarrow \mathrm{SUP}(S,T)$ by $(s\#t)(x) = t$ for $x \le s$ and $= 1$ otherwise.
Note that # is well-defined.

2.14. PROPOSITION. The function $\#: S^{op} \times T \longrightarrow \mathrm{SUP}(S,T)$ is a SUP-map. Its upper adjoint $U: \mathrm{SUP}(S,T) \to S^{op} \times T$ is given by $U(f) = (\sup f^{-1}(T \setminus \{1\}),\ f(0))$. Moreover, the function # is an INF-map in each variable separately. The lower adjoint of $a\#\cdot$ is $f \longmapsto f(a)$ , i.e. $f \le a\#x$ iff $f(a) \le x$, and the lower adjoint of $\cdot\#b$ is $f \longmapsto \hat{f}(b)$, i.e. $f \le x\#b$ iff $\hat{f}(b) \ge x$ , where $\hat{f}$ is the upper adjoint of $f$.

Proof. i) $s\#t \le f$ means $(x \le s \Rightarrow t \le f(x))$ and $(x \not\le s \Rightarrow 1 \le f(x))$, and this is equivalent to $f(\uparrow s) \subseteq \uparrow t$ and $f(S \setminus \downarrow s) = \{1\}$, which means $t \le f(0)$ and $f^{-1}(1) \supseteq S \setminus \downarrow s$ ,i.e. $\downarrow s \supseteq S \setminus f^{-1}(1) = f^{-1}(T \setminus \{1\})$ i.e. $\sup f^{-1}(T \setminus \{1\}) \le s$.

ii) $f \le a\#x$ means $f(\downarrow a) \subseteq \downarrow x$ ,i.e. $f(a) \le x$. Secondly, $f \le x\#b$ means $f(\downarrow x) \subseteq \downarrow b$, i.e. $f(x) \le b$, i.e. $x \le \hat{f}(b)$. $\square$

This must be linked with the tensor product à la Shmuely -Bandelt.