Gierz + Keimel – Stralka

SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME: Karl H.Hofmann	DATE	M	D	Y
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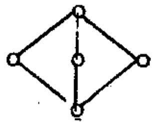
TOPIC: CL-projective limits of distributive continuous lattices are distributive

REFERENCES: Compendium IV-3
Gaskill,H.S. ,Classes of semilattices associated with an equational lattices,Can.J.Math. 25 (1973),332,338
Hofmann,K.H. and J.Thayer, Approximately finite dimensional C*-algebras, Dissert.Math. 1980, to appear.

Hofmann,K.H. et al. SCS 51 (30-5-79) and SCS 52 (11-6-79)

The spectral theory of C*-algebras led us to consider the category HL of continuous Heyting algebras with CL -morphisms as maps. Thus HL is a full subcategory of CL. The category HL is clearly closed in CL under the formation of arbitrary products, but it just as clearly fails to be closed under the simplest forms of finite limits, namely, subobjects: The non-distributive five element lattice



is obviously a subobject of 3^2, where 3 is the 3-element chain. Thus HL is not complete in CL. I believe it has escaped our attention that nevertheless HL is closed in CL under the formation of projective limits. I want to point this out here and state:

THEOREM. Within the category CL of continuous lattices and maps perserving arbitrary infs and directed sups, the full subcategory HL of all continuous Heyting algebras is closed under the formation of (arbitrary products and) projective limits.

Proof. a) The category S of all semilattices with identity and identity preserving morphisms has the property that the full subcategory S^d of distributive semilattices is closed under the formation of direct limits in S. This was shown by Gaskill,loc.cit., and I gave an independent proof in the paper on AFC*-algebras. By the duality of the category S with the category AL of algebraic lattices and CL -maps (Compendium p.184) and the fact that an algebraic lattice is an algebraic Heyting algebra iff the sup-subsemilattice K(L) is a distributive semilattice this observation shows that the assertion of the theorem holds for algebraic lattices.(In fact in the AFC*-paper it is pointed out,among other things, that an algebraic lattice is a Heyting algebra iff in AL it is a projective limit of finite distributive lattices.)

b) The ideal functor Id: CL $\longrightarrow$ AL preserves projective limits by Theorem IV-3.23 on p.221 of the compendium. Now let $(L_j, f_{jk};\ j,k \in J)$ be an inverse system in HL, and set $L = \lim L_j$ in CL.We must show that L is distributive. Now $(\mathrm{Id}\ L_j,\ \mathrm{Id}\ f_{jk},\ j,k \in J)$ is an inverse system in AL. Since all L_j are distributive, so are the Id L_j. By part a) above, lim Id L_j is distributive. But since Id preserves projective limits we have $\mathrm{Id}\ L \cong \lim \mathrm{Id}\ L_j$. Thus Id L and therefore L is distributive. Q.e.d.

In the light of the functorial spectral theory on HL which was detailed in the SCS-memo of 30-5-79 of myself and Watkins, where we show the equivalence of HL with the category LOC of locally quasicompact sober spaces and proper multi-valued maps mapping points to closed sets, it is then clear that the category LOC has projective limits. The techniques given in that memo suffice to calculate them explicitly once one knows the theorem above.