SEMINAR ON CONTINUITY IN SEMILATTICES (SCS) 56

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TOPIC On a question of O. Wyler

REFERENCE

1. G. Gierz et al.: A Compendium of Continuous Lattices
2. Hofmann K.H. and I. Watkins: A review of a theorem of Dixmier's, SCS-Memo 49 , 11-30-78
3. Wyler O.: On continuous lattices as topological algebras, Preprint
4. --- : Algebraic theories of continuous lattices

In his paper [3,4] , O. Wyler raises the following question: Let L be a lattice with 1 ; then the ideal lattice $\mathrm{Id}\,L$ is an algebraic lattice with an isolated greatest element. The set $\mathrm{Spec}\,\mathrm{Id}\,L$ of all proper prime ideals is then Lawson-closed, hence compact in the Lawson topology which equals the patch topology. We write $\mathrm{Spec}_p L$ when we consider the spectrum with the patch topology. Let $\Gamma(\mathrm{Spec}_p \mathrm{Id}\,L)$ be the continuous lattice of all closed subsets with respect to union. We define a map $D : \Gamma(\mathrm{Spec}_p \mathrm{Id}\,L) \longrightarrow \mathrm{Id}\,L$ by $D(C) = \bigcap C$ for any closed set C of prime ideals. Wyler notes that D is surjective iff L is distributive and asks:

QUESTION. When is D injective?

He observes:

REMARK. If L is a Boolean algebra, then D is injective.

In this Note we want to indicate that, for distributive lattices, the converse also holds.

In the following let S be any complete lattice and $\mathrm{Spec}\,S$ the set of all prime elements $p \neq 1$ of S . There is a natural map from the powerset of $\mathrm{Spec}\,S$ into S :

$$\begin{aligned} k: \mathcal{P}(\mathrm{Spec}\,S) &\longrightarrow S \\ A &\longmapsto \inf A \,. \end{aligned}$$

Conversely, there is a map from S onto the collection $\Gamma(\mathrm{Spec}\,S)$ of all hull kernel closed subsets of S :

$$\begin{aligned} h: S &\longrightarrow \Gamma(\mathrm{Spec}\,S) \subseteq \mathcal{P}(\mathrm{Spec}\,S) \\ a &\longmapsto \uparrow a \cap \mathrm{Spec}\,S \,. \end{aligned}$$

LEMMA 1. $k(A) = k(B)$ iff $h(k(A)) = h(k(B))$ i.e. iff A and B have the same hull kernel closure. □

CONSEQUENCE. Let λ be a topology on Spec S finer than the hull kernel topology and Γ_λ(Spec S) the collection of λ-closed sets:

(1) The restriction of k to Γ_λ(Spec S) is injective iff λ coincides with the hull kernel topology.

(2) If λ is T_1 and if the restriction of k to Γ_λ(Spec S) is injective, then the hull kernel topology is T_1 which implies that Spec S is an antichain. □

<u>LEMMA</u> 2. Let S be a continuous lattice such that Spec S $\cup \{1\}$ is Lawson-closed. Then the Lawson topology coincides with the hull kernel topology on Spec S iff Spec S is an antichain.

Proof. $\Longrightarrow$ is clear as the Lawson topology is Hausdorff on a continous lattice. $\Longleftarrow$: It suffices to show that, for every a in S, the set Spec S $\setminus \Uparrow a$ is hull kernel closed. For this we let b = inf A where A = Spec S $\setminus \Uparrow a$, and we show that h(b) = A. Indeed, let $q \in$ Spec S be such that $q \geq b$. By THE LEMMA there is an element s in the Lawson-closure of A with $q \geq s$. We have $s \in$ Spec S, as Spec S $\cup \{1\}$ is Lawson-closed. Thus q = s, as Spec S is supposed to be an antichain. As S$\setminus \Uparrow a$ is Lawson-closed, we finally have q = s $\in$ A. □

In a distributive continuous lattice, Spec S $\cup \{1\}$ is Lawson-closed iff $\ll$ is multiplicative. Thus we have:

<u>PROPOSITION</u>. For a distributive continuous lattice S, the following are equivalent:

(1) $\ll$ is multiplicative and every prime is maximal.

(2) Spec S $\cup \{1\}$ is compact in the Lawson topology λ and k : Γ_λ(Spec S) $\longrightarrow$ S is injective.

(3) S = O(X) for some locally compact Hausdorff space X. □

<u>COROLLARY</u>. For a distributive lattice L, the following are equivalent:

(1) D : Γ(Spec$_p$ Id L) $\longrightarrow$ Id L is injective.

(2) L is quasi-Boolean, i.e. every principal ideal of L is Boolean.

Indeed, a distributive lattice is quasi-Boolean iff every prime ideal is maximal.