SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)	M	D	Y
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TOPIC: On the duality of semilattices			
REFERENCES: Standard ; see also back of memo.			

1. Locally convex semilattices.

We consider semilattices which we tentatively call locally convex; in order not to commit ourselves too much we will simply speak of L semilattices.

1.1.DEFINITION . An L-semilattice is a semilattice S with identity 1 and with a topology O(S) which is linked with the semilattice structure as follows:

i) All $U \in O(S)$ are upper sets.

ii) O(S) has a basis of filters (i.e. $O(S) \cap$ Filt S is a basis of O(S)).

A morphism of L -semilattices is a continuous semilattice morphism preserving identities.

Of course L denotes the category of L -semilattices and L-semilattice morphisms. □

1.2.EXAMPLES. 1. Every semilattice with identity is an L-semilattice w.r.t. the indiscrete topology. This is not a particulalrly interesting one, and all the more interesting L-semilattices will have a T_o-topology, but for the moment we do not demand this.

2. Every semilattice with identity is an L-semilattice w.r.t. the Alexandroff-discrete topology $\alpha(S)$ which consists of all upper sets. This gives the finest L-semilattice structure which a semilattice with identity can carry. In particular, the distinguished semilattice $2 = \{0,1\}$ has one and only one T_o L-semilattice structure. We denote $(S, \alpha(S))$ with AS.

3. Let S be any up-complete semilattice and let $\varphi(S)$ be the topology generated by all Scott open filters. Then $(S, \varphi(S))$ is an L-semilattice. If S is continuous, then $\varphi(S) = \sigma(S)$ and the topology $\varphi(S)$ is T_o. We denote $(S, \varphi(S))$ with ΦS. □

1.3. REMARK. The assignment A is a functor from the category of all semilattices with identity into L, and Φ is a functor from the category of up-complete semilattices and Scott continuous semilattice morphisms preserving identities into L. Both are left adjoints for the respective grounding functors. □

Now let S be an L-semilattice. We consider the set $S^* = L(S,2)$ and observe that it is a subset of $2^{|S|}$. It is closed in the algebraic lattice $2^{|S|}$ under pointwise multiplication and pointwise directed sups. The Scott-topology of $2^{|S|}$ is generated by open filters, and hence so is the induced topology on S^*.

Thus if we equip S^* with the pointwise inf and the induced topology from the Scott topology of 2^S, then S^* is an L-semilattice. The topology on S^* is called $*$-topology. If $f:S \to T$ is a morphism of L-semilattices, then the morphism $f^*:T^* \to S^*$ of semilattices with identity which we define by $f^*(\chi) = \chi \circ f$ is continuous w.r.t. the $*$-topologies. Hence $*: L \to L$ is a contravariant functor.

1.4.REMARK. The category L is selfadjoint on the right via the functor $*$. Specifically, there is a natural isomorphism $L(S,T^*) \cong L(T,S^*)$ which is implemented as follows: If $f:S \to T^*$ is an arbitrary L-morphism from S to T^*, the the morphism $f': T \to S^*$ given by $f'(t)(s) = f(s)(t)$ is the unique one which makes the diagram

$$\begin{array}{ccc} S & \xrightarrow{\eta_S} & S^{**} \\ & {}_{f}\searrow & \downarrow f'^{*} \\ & & T^* \end{array}$$

commutative, where η_S is the front adjunction which is given by $\eta(s)(\chi) = \chi(s)$ for $\chi \in S^*$. □

This is fairly routine. Our interest will of course concentrate on specific properties of the dual S^* and on conditions on S which would ensure that η_S is an isomorphism. In this case we would say that S had duality.

1.5. EXAMPLES. 1. If S is an L-semilattice for which O(S) is indiscrete, then S^* is singleton.

2. If S is an arbitrary semilattice with 1, then $(AS)^*$, the dual of S w.r.t. the Alexandroff discrete topology is simply Hom(S,2), the character-semilattice of S in the sense of HMS-duality (Compendium p.184: S-AL-duality), and S^* may be canonically identified with Filt S, the semilattice of all filters.

3. If S is an up-complete semilattice, then $(\Phi S)^*$ is the Lawson dual of S which may be identified with the semilattice OFilt S of all (Scott-)open filters. (Compendium p.191.) □

We have seen in these examples and in many circumstances that it is essential to our intuition to have a geometric interpretation of the dual S^* as a semilattice of filters. This is what we discuss next.

1.6.PROPOSITION. Let S be an L-semilattice and OFilt S = $O(S) \cap$ Filt S. Then the assignment $\chi \mapsto \chi^{-1}(1): S^* \to$ OFilt S is an isomorphism whose inverse is given by $F \mapsto c_F$, where c_F is the characteristic function of F.

If $h:S^* \to$ OFilt S is this isomorphism, then for any element $s \in S$ we have $h^*\eta_S(s) = \{F \in \text{OFilt } S: x \in F\} \in$ OFilt OFilt S

$$\begin{array}{ccccc} & & S & & \\ & {}^{\eta_S}\swarrow & & \searrow{}^{h^*\eta_S} & \\ S^{**} & & \xrightarrow{h^*} & & \text{OFilt OFilt } S \end{array}$$

Proof straightforward.

1.7. DEFINITION. We will often identify S^* and OFilt S under the natural isomorphism of Proposition 1.6. The set $h^*\eta_S(s)$ is called the polar of s in $S^* = \mathrm{OFilt}\, S$ and is written s^o. The polar topology on S^* is the topology generated by all polars s^o, $s \in S$. □

1.8. PROPOSITION. The polar topology agrees on S^* with the topology established on the dual S^* as the topology induced from $2^{|S|}$.

Proof. The topology on $S^* = \underline{L}(S,2)$ is generated by the sets $U(s) = \{\chi \in S^* : \chi(s) = 1\}$. But $h(U(s)) = s^o$. □

The dual of an $\underline{L}$-semilattice is in many respects better behaved than the original semilattice.

1.9. PROPOSITION. The dual S^* of an L-semilattice S is up-complete and its topology is T_o.

Proof. A directed sup of open filters is a directed union of open filters and is, therefore an open filter. Thus OFilt S is up-complete. If $F \nleq G$, then there is an $s \in G \setminus F$ and then $G \in s^o$ and $F \notin s^o$. Thus the polar topology is T_o. □

1.10. PROPOSITION. Let S be an $\underline{L}$-semilattice. Then the following statements are equivalent:

(1) $\eta_S : S \longrightarrow S^{**}$ is injective.

(2) The characters in $\underline{L}(S,2)$ separate the points of S.

(3) The open filters of S separate the points, i.e. if $s \nleq t$ in S, then there is an open filter F with $s \in F$ and $t \notin F$.

(4) The topology $O(S)$ is T_o.

(5) $\{s\}^- = \downarrow s$ for all $s \in S$.

Proof. (1) ⇔ (2) is standard: Injectivity of η_S means that $\eta_S(s) = \eta_S(t)$ implies $s = t$; but $\eta_S(s) = \eta_T(t)$ means $\chi(s) = \chi(t)$ for all $\chi \in \underline{L}(S,2)$.

(2) ⇔ (3) is clear from Proposition 1.6.

(3) ⇔ (4) follows from the fact that OFilt S is a basis for $O(S)$. □

(4) ⇔ (5) is straightforward.

1.11. DEFINITION. An $\underline{L}$-semilattice S is called separated, if the equivalent conditions of 1.10 are satisfied, and the full subcategory in $\underline{L}$ of all separated $\underline{L}$-semilattices will be called $\underline{SL}$. □

1.12. PROPOSITION. Let S be an up-complete L-semilattice. Consider the following two conditions:

(1) The set IRR S of irreducibles separate the points of S (i.e. if $s \nleq t$, then there is a $p \in$ IRR S with $s \nleq p$ and $t \leq p$).

(2) S is separated.

Then (2) implies (1), and if S is a distributive semilattice, then (1) and (2) are equivalent if $O(S) = \varphi(S)$.

Proof. (2)⇒(1) Suppose $s \nleq t$. Since S is separated, there is an open filter U on S containing S, but not containing t. Since S is up-complete, by Zorn's Lemma there is a maximal element p above t in $S \setminus U$. This p is irreducible (Compendium p.79).

(1)⇒(2). Suppose that S is distributive. (Cf. Compendium p.77) Then every irreducible is prime. Thus if $s \nleq t$ we find a prime p with $s \nleq p$ and $t \leq p$. Then $U = S \setminus \downarrow p$ is a Scott-open filter, hence a $\varphi(S)$-open filter. This shows that $\varphi(S)$ is T_o. □

Recall: Condition (1) in Proposition 1.12 is equivalent to

(1') The set IRR S order generates S. (Cf. Compendium p.70) .

1.13. REMARK. Let S be an L-semilattice and R the equivalence relation $\{(x,y) : \text{For all } F \in \mathrm{OFilt}\, S \text{ we have } x \in F \text{ iff } y \in F\}$. Then R is a congruence, and in fact the kernel congruence of η_S. The quotient semilattice S/R is the universal T_o-quotient of S and the assignment $S \mapsto S/R$ is a left adjoint of the inclusion functor of $\underline{SL}$ into $\underline{L}$. □

2. Complete Heyting algebras with enough primes and coprimes.

2.1. DEFINITION. Let L be a cHa (complete Heyting algebra). Then cHa has enough primes iff Spec L separates the points, i.e. order generates L. Likewise we say that L has enough coprimes iff Cospec L = Spec L^{op} order-cogenerates L. If L has enough primes and coprimes, we say that L is bigenerated. □

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2.2. LEMMA. Let S be a separated L- semilattice. Then $O(S)$ is a cHa with enough primes and coprimes, i.e. $O(S)$ is bigenerated.

Proof. a) For each $s \in S$ the open set $S \setminus \{s\}^- = S \setminus \downarrow s$ is in Spec $O(S)$ as is the case for all topological spaces. Every open set is the intersection of these, whence Spec $O(S)$ order generates $O(S)$.

b) The proof, Compendium I-1.11 p.105, applies to show that $U \in O(S)$ is a coprime iff $U \in$ OFilt S. By 1.1.ii this means that Cospec $O(S)$ is order cogenerating. □

2.3. DEFINITION. We let bHa denote the category of bigenerated complete Heyting algebras with morphisms which preserve

i) arbitrary sups,

ii) finite infs,

iii) Cospecs.

2.4. REMARK. If $d : L \longrightarrow M$ is a map between bHa-algebras preserving arbitrary sups, and if $g : M \longrightarrow L$ is its upper adjoint, then d is a bHa map iff g preserves finite sups and Specs.

Proof. See Compendium p.188. □

2.5. REMARK. The assignment $S \mapsto O(S)$ is a contravariant functor $O : \underline{L} \longrightarrow$ bHa

Proof. This is clear from 2.2. □

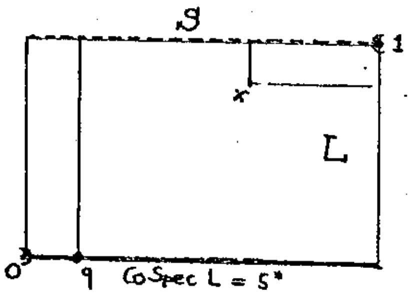
It is, of course, understood, that O acts on L-semilattices as on topological spaces.

Conversely, we make the following observation:

2.6.PROPOSITION. Let L be a bigenerated complete Heyting algebra. Suppose that S is an order generating subset of Spec L which $= (S, \not\geq)$ is closed in L under finite sups. Then, with the hull-kernel topology, S is a separated L-semilattice such that O(S) is isomorphic to L under the map $x \mapsto S \setminus \uparrow x : L \to O(S)$. Moreover, the map $q \mapsto S \setminus \uparrow q$: Cospec L $\longrightarrow$ OFilt S is an isomorphism which identifies Cospec L with the dual S^* of S.

Proof. By the spectral theory of continuous distributive lattices by Hofmann and Lawson we know that $x \mapsto S \setminus \uparrow x : L \to O(S)$ is an isomorphism of Heyting algebras and that the hull-kernel topology consists precisely of the sets $S \setminus \uparrow x$, $x \in L$. Since L is bigenerated, Cospec L order cogenerates L and this means that the hull kernel topology on S has a basis of sets $S \setminus \uparrow q$, $q \in$ Cospec L, and all of these are open filters in $(S, \geq)$ w.r.t. the hull kernel topology. Thus $S = (S, \geq)$ is an L-semigroup, and it is cearly separated since the hull-kernel topology is T_o. In order to complete the proof we have to show that whenever $S \setminus \uparrow q$ is a filter on $(S, \geq)$, then $q \in$ Cospec. But by 2.2, $S \setminus \uparrow q$ is a coprime of O(S), and by what we saw before $x \mapsto S \setminus \uparrow x : L \to O(S)$ is an isomorphism. Hence $q \in$ Cospec L. □

Thus the bigenerated Heyting algebras provide another way of looking at separated L-semilattices, and in fact a way which simultaneously exhibits the dual S^* in the same picture.



We continue the train of thought in Proposition 2.6 a bit further.

2.7.PROPOSITION. Under the conditions of Proposition 2.6, the *-topology on the dual S^* corresponds to the cospectral topology on Cospec L. If $s \in S$, then the polar corresponds to the set $\{q \in \text{Cospec } L : q \not\leq s\} =$ cohull s. The map $x \mapsto \text{Cospec } L \setminus \downarrow x : L^{op} \to O(\text{Cospec } L) \cong O(S^*)$ is an isomorphism of Heyting algebras.

Proof. From the definitions it follows directly that for $s \in S$, the polar s^o consists of all those filters $S \setminus \uparrow q$ containing s. The set of coprimes q which is so determined is precisely the cohull Cospec $L \setminus \downarrow s$. Also, by the spectral theory of Hofmann and Lawson, $x \mapsto \text{Cospec } L \setminus \downarrow x : L^{op} \longrightarrow O(\text{Cospec } L)$ is an isomorphism of Heyting algebras, where O(Cospec L) is the cospectral topology (the hull-kernel topology w.r.t. L^{op}) on Cospec L. Since every element of L is an inf of elements $s \in S$, then every open set in O(Cospec L) is a union of open sets of the form Cospec $L \setminus \downarrow s$ which correspond to the polars. Thus the Cospectral topology on Cospec L corresponds to the polar topology under the isomorphism Cospec $L \to S^*$. □

The symmetry of bigenerated Heyting algebras now allows us to identify Spec L as the bidual of S and hence as its sobrification.

2.8.THEOREM. Let L be a bigenerated Heyting algebra and S an order generating subset of Spec L which is closed under finite sups. Then Spec L is naturally isomorphic to the bidual S^{**} of the semigroup $(S, \geq)$ and the inclusion map corresponds to the front adjunction.

Proof. From Propositions 2,6 and 2.7 we know that Spec L may be identified with the dual (Cospec L)* of Cospec L under the isomorphism

$p \mapsto \text{Cospec } L \setminus \downarrow p$: Spec L $\longrightarrow$ OFilt Cospec L. Moreover, Cospec L may be identified with S^*, so that Spec L is naturally isomorphic to the bidual of S^*. The front adjunction $\eta_S : S \to S^{**}$ by 1.7 may be described in terms of the polar $s \to s^o$. The open filters of Cospec L correspond to the elements of Spec L under the map $p \mapsto \text{Cospec } L \setminus \downarrow p$. The open filter on Cospec L corresponding to s thus is Cospec $L \setminus \downarrow s$ which is precisely that set of coprimes which corresp[…] to the polar s^o under the identification of coprimes and open filters of S. Thus the front adjunction is equivalent to the inclusion map $S \to$ Spec L. □

We know from Lemma 2.2 that every separated L-semigroup is obtained as one of the semigroups S in the Theorem (with L = O(S).). Thus we immediately have a number of corollaries which directly impinge on the duality of L-semigroups.

2.9.COROLLARY. Let S be an L-semigroup. Then we have the following conclusions:

(i) S^* is sober in the *-topology.

(ii) $\eta_S : S \longrightarrow S^{**}$ is the sobrification of S.

(iii) S has duality iff it is sober.

Proof. First we observe that it is no loss of generality to assume that S is separated: The natural quotient $S \to S/R$ of 1.13 induces an isomorphism $(S/R)^* \longrightarrow S^*$ since every character $\chi : S \to 2$ factors through $S \to S/R$, since 2 is T_o. We thus have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\eta_S} & S^{**} \\ \downarrow & & \downarrow \cong \\ S/R & \xrightarrow[\eta_{S/R}]{} & (S/R)^{**} \end{array}$$

Moreover, the sobrification of S will always factor through $S \to S/R$ since sober spaces are T_o. Thus, from here on out we will assume that S is separated.

By Proposition 2.6 we know that S^* is isomorphic to Cospec L where $L = O(S)$. But spectra (hence cospectra) of complete Heyting algebras are sober. This shows (i). Next we know from the spectral theory of Hofmann and Lawson, that for each order generating subspace S in Spec L of a complete Heyting algebra L, the inclusion map $S \to \text{Spec } L$ is the sobrification map. Thus (ii) follows from 2.8. But (iii) is now an immediate consequence of (ii). □

Our Proposition 2.9 overlaps Proposition 1.10 of R.-E.Hoffman's in [3].

2.10. COROLLARY. If S is an L-semigroup, then S^* has duality, i.e. $\eta_{S^*}: S^* \to S^{***}$ is an isomorphism.

Proof. Immediate from 2.9. □

It may be useful to record a resumé:

2.11. SCHOLIUM. Let S be an L- semilattice.

A) The following statements are equivalent:

(1) S is sober.

(2) S is a dual (i.e. has a pre-dual, i.e. is of the form T^* for some L-semilattice T)

(3) S has duality (i.e. $\eta_S: S \to S^{**}$ is an isomorphism).

B) The duals in the category L (i.e. the objects satisfying (1),(2),(3) above span a full subcategory DL of L and $**: L \to DL$ is a left reflection (i.e. is left adjoint to the inclusion).

Proof. A) is just a summary of previous results.

B) Suppose that T is in DL and $f: S \to T$ is a morphism. Then there is a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\eta_S} & S^{**} \\ f \downarrow & & \downarrow f^{**} \\ T & \xrightarrow{\eta_T} & T^{**} \end{array}$$

in which η_T is an isomorphism. Thus $\eta_T^{-1} f^{**}$ is the required unique map $f': S^{**} \to T$ with $f = f' \eta_S$. Alternatively, we could have just as well invoked the fact that the sobrification is a left reflection. □

We wish to point out now that Example 1.2.3 has a much more general background.

2.12. PROPOSITION. Let N be the category of all semilattices with 1 together with a topology and morphisms being continuous semilattice maps preserving identities. (N stands for "nothing links topology and semilattice structure") For S in N we let $\varphi(S)$ denote the topology generated by $\text{OFilt } S = \text{Filt } S \cap O(S)$, and set

$\Phi S = (S, \varphi(S))$. Then Φ is a functor $N \to L$ and is indeed a left reflection. The front adjunction $j_S: S \to \Phi S$ is the identity map which induces an isomorphism

$$j_S^*: (\Phi S)^* \to S^*.$$

Proof. Let S be in N and T in L, and let $f: S \to T$ be an N-morphism. The topology of T is generated by OFilt T; if U is an open filter of T then $f^{-1}U$ is an open filter of S, hence is a member of $\varphi(S)$. Thus f factors through j_S. Thus Φ is a left reflection, as asserted. Since 2 is in L it follows that every character $\chi: S \to 2$ factors through j_S, hence j_S^* is bijective. □

Some additional information on *:

2.13. PROPOSITION. Let $f: S \to T$ be a surjective morphism of N-semigroups. Then $f^*: T^* \to S^*$ is an embedding.

Proof. We know that f^* is injective as a consequence of the surjectivity of f, and we know that it is continuous. We must show that is is relatively open. For this purpose we show that for any element $t \in T$ and any $s \in S$ with $f(s) = t$ (such s exists since f is surjective!) we have

(a) $s^o \cap \text{im}(\text{OFilt } f) = (\text{OFilt } f)(t^o)$

Take an open filter F in the left side of (a). Then $s \in F = f^{-1}G$ for some open filter G on T.. Then $t = f(s) \in G$, i.e. $G \in t^o$. Thus $F = f^{-1}G \in (\text{OFilt } f)(t^o)$.

Next take an open filter F in the right side of (a). This means $F = f^{-1}G$ with $t \in G$. Then $s = f(t) \in F$, i.e. $F \in s^o$, but also $F = f^{-1}G \in \text{im}(\text{OFilt } f)$.

Thus (a) is established. □

2.14. COROLLARY. Let S and T be two L-semilattices. Then any injective morphism $f: S \to T$ which induces a surjective dual $f^*: T^* \to S^*$ and for which T is sober, is an embedding.

Proof. By 2.11, $\eta_T: T \to T^{**}$ is an isomorphism. The map $\eta_S: S \to S^{**}$ is an embedding since S is separated, being injected into a sober space. By 2.13, $f^{**}: S^{**} \to T^{**}$ is an embedding. The commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \eta_S \downarrow & & \downarrow \eta_T \\ S^{**} & \xrightarrow{f^{**}} & T^{**} \end{array}$$

proves the claim.

3. - On the projective limit of sober spaces

R.-E.Hoffmann points out [4] that products of sober spaces are sober. We will show in this section that projective limits of sober spaces are sober.

One might surmise that in fact a stronger statement is true, namely, that the sobrification function $X \mapsto {}^sX$: TOP $\longrightarrow$ SOB from the category of topological spaces to the category of sober spaces preserves projective limits. The following example shows that this is not the case:

3.1. <u>EXAMPLE</u>. We let X_n, $n = 0,1,\dots$ be the set $\mathbb{N}_0 = \{0,1,2,\dots\}$ with the Scott topology. Then ${}^sX_n = \mathbb{N}_0 \cup \{\infty\}$ with a maximal element attached and with the Scott topology. The maps of the inverse system of the X_n are generated by the maps $f_{n-1,n}: X_n \longrightarrow X_{n-1}$, $n = 1,2,\dots$ given by $f_{n-1,n}(x) = 0$ for $x = 0$ and by $f_{n-1,n}(x) = x+1$, otherwise. Then $\lim X_n$ is the singleton $\{0\}$ with the limit maps f_n given by $f_n(0) = 0$. However, $\lim {}^sX_n = \{0, \infty\}$ with the limit maps sending 0 to 0 and ∞ to ∞.

This example also illustrates the fact, that the functor 0:TOP $\longrightarrow$ cHa^{op} does not transform projective limits into direct limits. □

In the proof we need a few facts, whose proof we leave to the reader:

3.2. <u>LEMMA.</u> For any continuous function $f: X \longrightarrow Y$ and any subset $A \subseteq X$ we have $f(A)^- = f(A^-)^-$. □

3.3. <u>LEMMA.</u> Let X be a space, A a subset of X and $f: X \longrightarrow Y$ a continuous function.

i) If A is irreducible (i.e. cannot be the union of two proper closed(in A) subsets of A), then A^- is irreducible.

ii) If A is irreducible, then f(A) is irreducible. □

3.4. <u>LEMMA</u>. Let X be the projective limit of an inverse system X_j of spaces in TOP and let $f_j: X \longrightarrow X_j$ be the limit maps. Then for any subset A of X we

$$A^- = \bigcap_j f_j^{-1}((f_jA)^-). \ □$$

Now we prove the following proposition:

3.5. <u>PROPOSITION</u>. Projective limits of sober spaces are sober.

Proof. Let $X = \lim X_j$ for a projective system of sober spaces, and let $f_j: X \to X_j$ be the limit maps. We take an irreducible closed set A in X and show the existence

of an $a \in X$ with $A = \{a\}^-$.

Firstly, by Lemma 3.3, $f_j(A)^-$ is a closed irreducible set in X_j for all j. Since X_j is sober, we obtain an $a_j \in X_j$ with $f_j(A)^- = \{a_j\}^-$.
Secondly we claim that there is an $a \in X$ with $f_ja = a_j$ for all j. For a proof of this claim we have to show that $f_{ij}a_j = a_i$ for all $i < j$. But now we have $\{a_i\}^- = (f_i A)^- = (f_{ij}f_jA)^- = (f_{ij}(f_jA)^-)^-$ (by Lemma 3.2) $= (f_{ij}\{a_j\}^-)^-$ $= \{f_{ij}a_j\}^-$ (by Lemma 3.2), and since the dense point in a singleton closure in a sober space is unique, the claim $a_i = f_{ij}a_j$ and thus the existence of the wanted a is established.

Thirdly, we finish the proof by showing $A = \{a\}^-$. According to Lemma 3.4, we have $A = \bigcap_j f_j^{-1}(f_jA)^-$, but this is equal to $\bigcap_j f_j^{-1}(\{a_j\}^-)$ $= \bigcap_j f_j^{-1}(\{f_ja\}^-) = \bigcap_j f_j^{-1}(f_j(\{a\}^-)^-)$ (by Lemma 3.2); and by Lemma 3.4 again the last expression is equal to $\{a\}^-$. □

<u>COROLLARY</u> 3.6. The category <u>DL</u> of <u>L</u>-semilattices with duality is closed in <u>L</u> under the formation of projective limits.

Proof. This is a consequence of 2.9.ii and 3.5 above

4. More about bigenerated complete Heyting algebras.

We need more information on bigenerated Heyting algebras. We start with a more general situation.

4.1 . LEMMA. Let L be a lattice which is order cogenerated by $P =$ Cospec L. Let $p \in$ Spec L. Then

(i) $U = P \setminus \downarrow p$ is a filter on P.

(ii) $\uparrow U = L \setminus \downarrow p$.

Proof. (i) Clearly U is an upper set in P. Let $a, b \in U$. Then $a, b \notin \downarrow p$ and so $ab \notin \downarrow p$, since p is prime. Because of $ab = \sup \downarrow ab \cap P$, there must be a $q \in P$ with $q \notin \downarrow p$ and $q \leqslant ab \leqslant a$ and b. But $q \in P \setminus \downarrow p = U$.

(ii) Since $U = P \setminus \downarrow p$, clearly $\uparrow U \subseteq L \setminus \downarrow p$. Now let $x \in L \setminus \downarrow p$. Since $x = \sup \downarrow x \cap P$, there is a $q \in P$ with $q \notin \downarrow p$, and $q \leqslant x$. But $q \in P \setminus \downarrow p = U$, and so $x \in \uparrow U$. □

4.2. LEMMA. Under the hypotheses of Lemma 2.15, $P \setminus \downarrow p = P \setminus \downarrow p'$ implies $p = p'$ for any $p, p' \in$ Spec L.

Proof. Suppose $p \neq p'$, say $p \not\leqslant p'$. Since $p = \sup \downarrow p \cap P$, there is a $q \in P$ with $q \not\leqslant p'$ but $q \leqslant p$. This is a contradiction to $P \setminus \downarrow p = P \setminus \downarrow p'$. □

This leads us to the following observation:

4.3. PROPOSITION. On any complete lattice which is order cogenerated by its cospectrum Cospec L, there is a canonical injection of up-complete posets

$$\pi_L : (\text{Spec } L, \geqslant) \longrightarrow \text{OFilt}(\text{Cospec } L, \leqslant), \quad \pi_L(p) = (\text{Cospec } L) \setminus \downarrow p.$$

This injection is Scott continuous.

Proof. Lemmas 2.15 and 2.16 show that π_L is well-defined and injective. If $\{p_j\}$ is a filtered family of primes, then $p = \inf p_j$ exists in Spec L, and $\downarrow p = \bigcap \downarrow p_j$. Then $\pi_L(p) = P \setminus \downarrow p = \bigcup (P \setminus \downarrow p_j) = \sup \pi_L(p_j)$. □

The question is when is π_L surjective? Whenever U is an open filter in $P =$ Cospec L, we consider the set $\uparrow U$. For any filter F on P, its upper set $\uparrow F$ in L is a prime filter, i.e. $L \setminus \uparrow F$ is an ideal. Thus if $\uparrow U$ Scott open in L, then $p = \max L \setminus \uparrow U$ is a prime in L and $\pi_L(p)$ has a chance to be the required U. We formalize that as follows:

4.4 . PROPOSITION. Let L be a complete lattice which is cogenerated by its cospectrum. Then the following conditions are equivalent:

(1) π_L is an isomorphism of posets.

(2) For each Scott open filter U in Cospec L the set $\uparrow U$ is Scott open in L.

(3) The function $U \mapsto U \cap$ Cospec L maps the set of Scott open prime filters of L onto the set of Scott open filters of Cospec L.

Moreover, if these conditions are satisfied, then $\pi_L^{-1}(U) = \max L \setminus \uparrow U$.

Proof. Since $\uparrow F$ is a prime filter of L for every filter of $P =$ Cospec L, it is clear that (2) implies (3). We claim that (3) $\Rightarrow$ (2): First we establish the following claim: If U and V are open prime filters of L then $U \cap P = V \cap P$ implies $U = V$. Indeed we show more accurately that for an open prime filter U and a prime filter V with $U \setminus V \neq \emptyset$ we have $(U \cap P) \setminus V \neq \emptyset$: Now suppose that $U \setminus V \neq \emptyset$, say $x \in U \setminus V$. Since $x = \sup \downarrow x \cap P$ and since U is open, there is a finite subset $F \subseteq \downarrow x \cap P$ with $\sup F \in U$; since U is prime, there is a $q \in F$ with $q \in U$; but since $q \leqslant x$ we cannot have $q \in V$; thus $q \in (P \setminus U) \cap (P \setminus V)$.

Thus condition (3) in fact is equivalent to

(3') The function $U \mapsto U \cap P$ maps the set of open prime filters of L bijectively onto the set of open filters of P.

If this condition is satisfied, the for every open filter U of P there is one and only one prime filter U' of L with $U = U' \cap P$. Clearly $\uparrow U \subseteq U'$. Since U' is an open prime filter and $\uparrow U$ is a prime filter, the argument above shows that $\uparrow U = U'$ since $U' \cap P = U = \uparrow U \cap P$.

Thus (2) and (3) are equivalent.

For each open prime filter U of L the set $L \setminus \uparrow U$ is a closed ideal, whose maximal element is a prime p, and conversely, if $p \in$ Spec L, then $L \setminus \downarrow p$ is an open prime filter. Thus $p \mapsto L \setminus \downarrow p : (\text{Spec } L, \geqslant) \longrightarrow$ (open prime filters of $L, \subseteq$) is an order isomorphism. In view of this fact, (1) and (3') are clearly equivalent, and if these conditions are satisfied, then $\pi_L^{-1}(U) = \max L \setminus \uparrow U$ follows.

4.5 . DEFINITION. We say that a complete lattice L is strongly cogenerated iff it is cogenerated by its cospectrum and the equivalent conditions of Proposition 4.4 are satisfied.

4.6 . REMARK. If S is an L-semigroup. Then $O(S)$ is strongly cogenerated iff the following condition is satisfied:

If $\mathcal{F}$ is a filterbasis on S whose members are open filters of S and which is Scott open in the semilattice of open filters on S, then the filter $\mathcal{U}$ of open sets generated by $\mathcal{F}$ on S is Scott open in $O(S)$.

4.7 . COROLLARY. Suppose that S is an L-semigroup. Then the following conditions are equivalent:

(1) $O(S)$ is strongly cogenerated (see 4.5).

(2) The open filters on S^* and the Scott open filters on S^* agree.

Proof. We let $L = O(S)$; then S may be identified with an order generating subset of Spec $L = L^{**}$ and S^* may be identified with Cospec L. The open filters of S^* correspond bijectively to the elements $p \in$ Spec L via π_L: Spec $L \to$ OFilt*Cospec L where Ofilt* Cospec L is the semilattice of *-open filters on Cospec L = S*. Now condition (1) is equivalent to

(1') π_L: Spec L $\longrightarrow$ OFilt$^{\sigma}$Cospec L is bijective

by 4.4 and 4.5, and (2) is equivalent to

(2') OFilt* Cospec L = OFilt$^{\sigma}$ Cospec L.

But now it is clear that (1') and (2') are equivalent. □

Let us remark that among all filters F of L the ones of the form $L\setminus\downarrow p$ for a prime p are precisely the completely prime filters.

4.8'. DEFINITION. Let L be a lattice and F a filter. A $p\in$ Spec L is called a limit point of F iff $L = \downarrow p \cup F$, and lim F is the set of all limit points.

4.9. LEMMA. For a lattice L and a filter F the following statements are equivalent:

(1) F is completely prime (i.e. max $L\setminus F$ exists in Spec L and $F = L\setminus\downarrow p$.)
(2) lim F has a smallest element.
(3) lim F is filtered.
(4) For each $x\in F$ there is a $p\in$ lim F with $x\not\leq p$.

Proof. (1)⇔(2) is clear from the definitions, and (2)⇒(3) is trivial. If (3) holds, then inf lim F is a prime p. The following Lemma shows that $p\in$ lim F. So (2)⇔(3). (1)⇒(4) is clear. If (4) is satisfied, then (in view of 4.10 below) lim F is a closed

rreducible ... t in ... ec L ... nce ... tisfies ... 2). □

4.10. LEMMA. lim F is hull-kernel closed.

Proof. Let $p\in(\lim F)^{-}$. We must show $L\setminus\downarrow p\subseteq F$. Let $x\not\leq p$; then, since $p\in(\lim F)^{-}$ there is a $q\in$ lim F such that $x\not\leq q$. But $\downarrow q\cup F = L$, whence $x\in F$. □

These concepts allow us yet another equivalent formulation of the conditions in 4.4.

4.11. PROPOSITION. Let L be a complete lattice which is cogenerated by its cospectrum Then the equivalent conditions of 4.4 are also equivalent to

(4) For each open filter U on Cospec L and each $u\in U$ there is a $p\in \lim\uparrow U$ with $u\not\leq p$.

Proof. Since Cospec cogenerates L, condition (4) above is equivalent to conditon (4) of Lemma 4.9 Thus (4) above is equivalent to the condition that for all open filters U of Cospec L the filter U is completely prime, and this is equivalent to (1) of 4.4. □

4.12. COROLLARY. Assume that, under the conditions of 4.11, S is an order generating subset of Spec L. Then (4) in 4.11 is also equivalent to

(4') For each open filter U on Cospec L and each $u\in U$ there is a $p\in(\lim\uparrow U)\cap S$ with $u\not\leq p$. □

We obtain sufficient conditions for the validity of (1)-(4) in 4.4 and 4.11. These conditions will play an important role in the next section.

4.13. DEFINITION. We say that an element s in a poset S = Cospec L (for a distributive lattice L) has the refinement property, if s = sup X in L for a set $X\subseteq S$ entails the existence of a directed set $D\subseteq\downarrow X\cap S$ with s = sup D. □

4.14. PROPOSITION. Let L be a complete lattice which is cogenerated by its cospectrum. Then conditions (1)-(4) of 4.4 and 4.11, and, in case Spec L contains an order generating subset S, condition (4') of 4.12 are satisfied provided that the following hypothesis holds:

(r) Every point of Cospec L has the refinement property.

Proof. We verify (4'). Suppose that (4') does not hold. Then there is an open filter U of Cospec L and a $u\in U$ such that for all $p\in S\cap\lim\uparrow U$ one has $u\leq p$, i.e. that $S\cap\lim\uparrow U\subseteq\uparrow u$. In other words, $(S\setminus\uparrow u)\cap\lim\uparrow U=\emptyset$. Thus no $x\in S\setminus\uparrow u$ can be a limit point of U; hence there is a $u_x\not\leq x$ with u_x; we may assume $u_x\leq u$. We claim that $\sup\{u_x : x\in S\setminus\uparrow u\} = u$, for otherwise one would find a $y\in S\setminus\uparrow u$ with $\sup u_x\leq y$; then in particular $u_y\leq y$, which is impossible.

By (r) there is a directed set $D\subseteq\downarrow\{u_x : x\in S\setminus\uparrow u\}$ with sup D = $u\in U$. Since U is open, there is a $d\in D$ with $d\in U$; then there is an $x\in S\setminus\uparrow u$ with $d\leq u_x$, which implies $u_x\in U$, a contradiction. □

4.16. LEMMA. Suppose that $U\in$ OFilt S has a countable basis (i.e. a countable subset C with $U = \uparrow C$). Then U has the refinement property in OFilt S = Cospec S .

Proof. We may assume that $C = \{c_n : n = 1,2,\ldots\}$ with $c_{n+1}\leq c_n$. Let $\mathcal{U}$ be a collection of open filters with $U = \bigcup\mathcal{U}$. For each n we pick a U_n with $c_n\in U_n$ and set $V_n = \bigcap\{U_m : n\leq m\}$. Then V_n is a filter; we show that it is open: Let D be a directed set with sup $D\in V_n$. Then sup $D\in U$ and since U is open, there is a $d\in D$ with $d\in U = \bigcup\uparrow c_k$. Hence there is an N such that $c_N\leq d$. For each k with $n\leq k<N$ there is a $d_k\in D$ with $d_k\in U_k$, since U_k is open. As D is directed, we find an $e\in D$ with $d\leq e$ and $d_k\leq e$ for all k with $n\leq k<N$. Then $e\in U_k$ with $n\leq k<N$ and $e\in\uparrow c_m\subseteq U_m$ for $m\geq N$. Hence $e\in V_n$. This shows that V_n is open. Now we have $V_n\subseteq V_{n+1}$ and $U = \bigcup\uparrow c_n\subseteq\bigcup V_n\subseteq U$. Clearly $V_n\in\downarrow\mathcal{U}$. □

We will utilize this Lemma in Section 7.

5. Duality for up-complete semilattices.

5.1. DEFINITION. We denote with $\underline{U}$ the category of all up-complete semilattices with identity and Scott continuous identity preserving semilattice morphisms. For S in $\underline{U}$ we define

$S^{\wedge} = \underline{U}(S,2)$ with the semilattice structure induced from $2^{|S|}$.

$\mathrm{OFilt}\ S = \mathrm{Filt}\ S \cap \sigma(S)$. □

Note that $S^{\wedge}$ is well-defined since $\underline{U}(S,2)$ is multiplicatively closed in $2^{|S|}$.

5.2. PROPOSITION. i. For S in $\underline{U}$, the function $\chi \mapsto \chi^{-1}(1): S^{\wedge} \longrightarrow \mathrm{OFilt}\ S$ is an isomorphism of semilattices.

ii. $S^{\wedge}$ is up-complete and $^{\wedge}: \underline{U} \longrightarrow \underline{U}$ is a contravariant functor which is an adjoint on the right. The front adjunction $\varepsilon_S: S \longrightarrow S^{\wedge\wedge}$ is defined by $\varepsilon_S(s)(\chi) = \chi(s)$ (resp., by $s \mapsto s^o: S \longrightarrow \mathrm{OFilt}\ S$ defined by $s^o = \{F \in \mathrm{OFilt}\ S:\ s \in F\}$).

Proof. Exercise. □

We would like to see how $^{\wedge}$ and $*$ relate.

5.3. REMARK. Let S be an up-complete semilattice. Then $S^{\wedge}$ is the underlying semilattice of $(S, \varphi(S))^*$. □

Here $\varphi(S)$ is the filter topology associated with the Scott topology $\sigma(S)$, i.e. the topology generated by OFilt S. We will denote the $\underline{L}$-semilattice $(S, \varphi(S))$ by ΦS.

5.4. PROPOSITION. Let S be an up-complete poset. Then the following conditions are equivalent:

(1) The equivalent conditions of 1.10 hold for ΦS.

(2) $\varepsilon_S: S \longrightarrow S^{\wedge\wedge}$ is injective.

Note that these conditions say that the Scott open filters separate the points. □

5.5. PROPOSITION. Let S be an up-complete semilattice. Then the identity function $i_S: \Phi(S^{\wedge}) \longrightarrow (\Phi S)^*$ is continuous and induces an embedding

$$i_S^*: (\Phi S)^{**} \longrightarrow (\Phi S^{\wedge})^*.$$

Proof. The topology of $(\Phi S)^*$ is generated by the polars s^o (where we identify $(\Phi S)^*$ and OFilt S). These polars are Scott open filters on $\mathrm{OFilt}\ S = S^*$. Hence they are open on $\Phi(S^{\wedge}) = (S^{\wedge}, \varphi(S^{\wedge}))$. The remainder then follows from 2.13. □

We say that an up-complete semilattice S has duality iff $\varepsilon_S: S \longrightarrow S^{\wedge\wedge}$ is an isomorphism.

5.6. THEOREM. For an up-complete poset S, the following conditions are equivalent:

(1) S has duality.

(2) ΦS is sober and $i_S: \Phi(S^{\wedge}) \longrightarrow (\Phi S)^*$ is a homeomorphism.

Proof. We consider the following commutative diagram

$$\begin{array}{ccc} \Phi S & \xrightarrow{\ \Phi\varepsilon_S\ } & \Phi S^{\wedge\wedge} \\ \downarrow \zeta_{\Phi S} & & \downarrow i_{S^{\wedge}} \\ (\Phi S)^{**} & \xrightarrow[\ i_S^*\ ]{} & (\Phi S^{\wedge})^* . \end{array}$$

We know that the right vertical arrow is a bijection and the lower horizontal one an embedding by 5.5. The other two arrows are injective precisely when the open flters separate points on S.

If S has duality, then the top horizontal arrow is an isomorphism, hence the diagonal $i_{S^{\wedge}}(\Phi\varepsilon_S) = i_S^*(\zeta_{\Phi S})$ is bijective. Hence i_S^* must be surjective and hence an isomorphism, whence $\zeta_{\Phi S}$ is an isomorphism, too. This means that ΦS is sober by 2.9. We have to show that i_S is open: If we take a Scott open filter on $S^{\wedge}$, then since i_S^* is surjective, this filter is also $*$-open. Hence i_S is open since the open filters generate the respective topologies.

Conversely, suppose that condition (2) is satisfied. Since ΦS is sober, the left vertical arrow is an isomorphism by 2.9. Since i_S is an isomorphism, then the lower horizintal arrow is an isomorphism. Thus the diagonal is an isomorphism, and since $i_{S^{\wedge}}$ is bijective, it follows that ε_S must be bijective. □

We have actually shown the following additional information.

5.7. REMARK. If the conditions of Theorem 5.6 are satisfied, then

$$i_{S^{\wedge}}: \Phi S^{\wedge\wedge} \longrightarrow (\Phi S^{\wedge})^*$$

is a homeomorphism, i.e. the $*$-topology and the filter topology agree on $\Phi S^{\wedge}$. □

It is perhaps useful to isolate once more the condition involving i_S:

5.8. PROPOSITION. Let S be an up-complete poset. Then the following conditions are equivalent, where $L = \varphi(S)$:

(1) $i_S: \Phi(S^{\wedge}) \longrightarrow (\Phi S)^*$ is a homeomorphism.

(2) On Cospec L the cospectral topology agrees with the filter topology associated with the Scott topology of Cospec L.

(3) The open filters of $(\Phi S)^*$ and the Scott open filters of $S^{\wedge}$ agree.

(4) $\varphi(S)$ is strongly cogenerated (see 4.5').

(5) i_S^* is surjective.

Proof. Under the identification of $(\Phi S)^*$ with Cospec L, the $*$-topology corresponds to the cospectral topology (induced by the upper topology $\upsilon(L)$). Then (1) ⇔ (2) is clear. (3) is a translation of (2). And (3) ⇔ (4) is 4.7. (3) is equivalent to (5) (see diagram in 5.6). □

5.9. COROLLARY. Let S be an up-complete poset. Then (1)-(5) of 5.8 are also equivalent to

(6) $\varepsilon_S : S \longrightarrow S^{\wedge\wedge}$ is the sobrification map of ΦS with the specialisation order considered on its domain and codomain.

Proof. Recall that (5) is equivalent to (5'): $i_S{}^*$ is a homeomorphism. The equivalence of (6) and (5) is then clear from the diagram in 5.6. □

Let us observe some functorial aspects of the assignment $S \mapsto \varphi(S)$.

5.11 PROPOSITION. The assignment $S \mapsto \varphi(S)$ is a contravariant functor

$$\varphi : \underline{U} \longrightarrow \text{bHa} \quad (\text{see } 2.3)$$

(with $\varphi(f)(V) = f^{-1}V$ for $f : S \longrightarrow T$ as usual). If $S^\wedge$ is identified with Cospec $\varphi(S)$, then f^* becomes identified with $\varphi(f)|\text{Cospec } T$ for $f : S \longrightarrow T$ in $\underline{U}$. □

5.10. COROLLARY. Let S be an up-complete poset. Then (1)-(6) of 5.8 and 5.9 are also equivalent to the following condition:

(7) Let $\mathcal{U}$ be an Scott open filter on OFilt S and let $U \in \mathcal{U}$. Then U contains a limit point of $\mathcal{U}$.

Proof. We apply 4.12 with $L = \varphi(S)$. Then condition (4') of 4.12 is equivalent to condition (7) above. Then 4.12 proves the claim. □

6. Duality of projective limits of up-complete semilattices

We recall the important diagram from the proof of 5.6:

6.1. RECALL. For an up-complete semilattice S in U we have a commutative diagram

$$(1)\qquad \begin{array}{ccc} \Phi S & \xrightarrow{\Phi\varepsilon_S} & \Phi S^{\wedge\wedge} \\ \eta_{\Phi S}\downarrow & & \downarrow i_{S^\wedge} \\ (\Phi S)^{**} & \xrightarrow{i_S{}^*} & (\Phi S^\wedge)^* \end{array}$$

in which $i_{S^\wedge}$ is bijective and $i_S{}^*$ an embedding. $\eta_{\Phi S}$ is an isomorphism iff ΦS is sober. In this case, ε_S is an isomorphism iff $i_S{}^*$ is surjective, and for this we have the 7 equivalent conditions of 5.8, 5.9 and 5.10. □

In this section we will consider an inverse system S_j of $\underline{U}$-objects with their projective limit $S = \lim S_j$ in $\underline{U}$. The limit maps are denotes $f_j : S \to S_j$

6.2. DEFINITION. If $T = \lim \Phi S_j$ is the projective limit of the inverse system ΦS_j in $\underline{L}$, then the unique morphism $S \longrightarrow T$ in the diagram

$$\begin{array}{ccc} S = \Phi(\lim S_j) & \longrightarrow & \lim \Phi S_j = T \\ & {\scriptstyle \Phi f_j}\searrow \quad \swarrow{\scriptstyle g_j} & \\ & \Phi S_j & \end{array}$$

with the limit maps $g_j : T \longrightarrow S_j$ will be denoted $\alpha : \Phi S \longrightarrow T$.

This morphism will play a crucial role, as we see in the following theorem:

6.3. THEOREM. Let $S = \lim S_j$ be a projective limit in $\underline{U}$ and suppose that the following two conditions are satisfied:

i) $\varepsilon_{S_j} : S_j \longrightarrow S_j{}^{\wedge\wedge}$ is an isomorphism for all j (i.e. all S_j have duality).

ii) $\alpha : \Phi(\lim S_j) \longrightarrow \lim(\Phi S_j)$ is an isomorphism.

I.e., the filter topology associated with the Scott topology of $\lim S_j$ is the limit topology of the filter topologies of the S_j.

Then $\varepsilon_S : S \longrightarrow S^{\wedge\wedge}$ is an isomorphism, i.e. S has duality.

Proof. We apply diagram (I) above to S_j; from i) we know that $i_{S_j^\wedge}(\Phi\varepsilon_{S_j})$ is biject... whence $i_{S_j}{}^*$ is surjective, hence an isomorphism. Then $\eta_{\Phi S_j}$ is necessarily biject... Now we can apply Theorem 3.6 and conclude that

$\eta_{\lim \Phi S_j} : \lim \Phi S_j \longrightarrow (\lim \Phi S_j)^{**}$ is an isomorphism. Hypothesis ii) then tells us that

$\eta_{\Phi(\lim S_j)} : \Phi(\lim S_j) \longrightarrow \Phi(\lim S_j)^{**}$ is an isomorphism.

From diagram (I), applied to $S = \lim S_j$ we observe that, in order to prove the claim we have to show that

(1) $i_{\lim S_j}{}^{*} : (\Phi \lim S_j)^{**} \longrightarrow (\Phi(\lim S_j)\hat{\ })^{*}$ *is surjective.*

is surjective. In order to prove this claim, we consider the diagram

(II)
$$\begin{array}{ccccc}
(\Phi \lim S_j)^{**} & \xrightarrow{\ i_{\lim S_j}{}^{*}\ } & & & (\Phi(\lim S_j)\hat{\ })^{*} \\
 & \searrow^{(\Phi f_j)^{**}} & & & \searrow^{(\Phi \hat{f}_j)^{*}} \\
\alpha^{**}\ \big\downarrow \cong & & (\Phi S_j)^{**} \ \cong & \big\downarrow \beta & \longrightarrow (\Phi S_j\hat{\ })^{*} \\
 & \nearrow & & & \nearrow \\
(\lim (\Phi S_j))^{**} & & \xrightarrow{\ \gamma\ } & & \lim((\Phi S_j\hat{\ })^{*})
\end{array}$$

where β is the natural map given by the limit. Also γ is induced by the limit, and since $i_{S_j}{}^{*} : (\Phi S_j)^{**} \longrightarrow (\Phi S_j\hat{\ })^{*}$ is an isomorphism, it follows that γ is an isomorphism, too.

We now notice that condition (1) is equivalent to the following

(2) $i_{\lim S_j}{}^{*}$ is bijective.

(3) β is injective.

(4) β is bijective.

(5) The family of maps $(\Phi f_j)^{*}$ separates the points of $(\Phi \lim S_j)\hat{\ })^{*}$.

Indeed (1) $\Leftrightarrow$ (2) by 6.1, (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) from the rectangle in (II), and (3) $\Leftrightarrow$ (5) is a basic and elementary fact on limits. For the following discussion we recall the abbreviation $T = \lim S_j$. Now we reformulate (5):

(6) For any pair a,b of characters in $(\Phi T\hat{\ })^{*}$, the relations $(\Phi f_j\hat{\ })(a) = (\Phi f_j\hat{\ })(b)$ for all j imply a= b.

In order to understand this condition we have to analyze the expression $(\Phi f_j\hat{\ })(a)$, which is an element of $(\Phi S_j\hat{\ })^{*}$. Its definition is given in the diagram

$$\begin{array}{ccc}
\Phi S_j\hat{\ } & \xrightarrow{\ \Phi f_j\hat{\ }\ } & \Phi T\hat{\ } \\
 & {\scriptstyle (\Phi f_j\hat{\ })^{*}(a)} \searrow \quad \swarrow {\scriptstyle a} & \\
 & 2^{\cdot} &
\end{array}$$

Next we have to be able to evaluate $(\Phi f_j\hat{\ })$ at a character $\chi \in \hat{S}_j$. Here we have a similar diagram:

$$\begin{array}{ccc}
T & \xrightarrow{\ f_j\ } & S_j \\
 & {\scriptstyle (f_j\hat{\ })(\chi)} \searrow \quad \swarrow {\scriptstyle \chi} & \\
 & 2^{\cdot} &
\end{array}$$

Thus $f_j\hat{\ }(\chi) = \chi \circ f_j = \chi \circ f_j$.
Now we can reformulate the equality $(\Phi f_j\hat{\ })^{*}(a) = (\Phi f_j\hat{\ })^{*}(b)$ as

(*) $a(\chi \circ f_j) = b(\chi \circ f_j)$ for all $\chi \in S_j\hat{\ }$.

We now get a better feeling if we translate this statement from characters into open filters. The $T\hat{\ }$ is identified with OFilt T (the semilattice of Scott-open filters on T), further S_j is identified with OFilt S_j; then χ becomes replaced by the open filter $V = \chi^{-1}(1)$ and $\chi \circ f_j$ by $f_j^{-1}V$. We then reformulate (6) once more

(7) For any pair of characters $a,b \in$ (OFilt T)*, the relations
$a(f_j^{-1}V) = b(f_j^{-1}V)$ for all j and all $V \in$ OFilt S_j imply
$a(U) = b(U)$ for all $U \in$ OFilt T .

Now the family of all $f_j^{-1}V$, where j ranges through all indices and V through all open filters of S_j is a basis B for a topology on $T = \lim S_j$, namely the basis f the limit topology of $\lim \Phi S_j$. The filter topology of the Scott topology of $T = \lim S_j$ is the topology of ΦT. Thus we can reformulate condition (7) once again:

(8) If two characters in Φ(OFilt T)* (i.e. two filter topology continuous characters of OFilt T) agree on the basis B of $O(\lim \Phi S_j)$ then they agree on the basis OFilt T of $O(\Phi T) = \varphi(T)$, $T = \lim S_j$.

Now we recall that $B \subseteq$ OFilt T and that therefore $O(\lim \Phi S_j) \subseteq O(\Phi T)$. However, by Hypothesis ii) of the Theorem we have $O(\lim \Phi S_j) = O(\Phi T) = \varphi T$. Thus B is also a basis of ΦT. In particular, every element of OFilt T is a directed sup of elements in B. However, a φ-continuous character on an up-complete semilattice is Scott continuous. (See 5.5: bijectivity of i_S). Thus the characters considered in (8) preserve directed sups, and this condition (8) is satisfied. This completes the proof.

6.4. COROLLARY. Let S_j be an inverse system in $\underline{U}$ and let $S = \lim S_j$ in $\underline{U}$. Assume that

i) all S_j have duality (i.e. $\varepsilon_{S_j}: S_j \longrightarrow S_j\hat{}\hat{}$ is an isomorphism for all j)

ii) on all S_j the Scott topology is generated by its open filters (i.e. $\varphi(S_j) = \sigma(S_j)$ for all j)

iii) The limit topology on $\lim S_j$ w.r.t. the Scott topologies on S_j is generated by its open filters (i.e. $O(\lim \Sigma S_j) = \varphi(\lim S_j)$)

Then S has duality.

Proof. Immediate from the previous Theorem. □

6.5. COROLLARY. Under otherwise unchanged conditions in 6.4, in lieu of condition iii) the following condition is sufficient:

iii') The Scott of the limit is the limit of the Scott. (I.e. $\sigma(\lim S_j) = O(\lim \Sigma S_j)$).

Proof. By ii) we have $O(\lim \Sigma S_j) = O(\lim \Phi S_j) \subseteq \varphi(\lim S_j) \subseteq \sigma(\lim S_j)$. Hence (iii') implies (iii). □

6.6. COROLLARY. Let S_j be an inverse system of continuous semilattices with $S = \lim S_j$. If $O(\lim \Sigma S_j) = \varphi(\lim S_j)$, then S has duality. This is satisfied in particular if $O(\lim S_j) = \sigma(\lim S_j)$.

Proof. Continuous semilattices have duality, and their Scott topology is generated by open filters. □

6.7. COROLLARY. Let S_j be an inverse system of continuous semilattices with $S = \lim S_j$. Suppose that each of the limit maps $f_j: S \longrightarrow S_j$ has a Scott lower adjoint $d_j: S_j \longrightarrow S$. Then S has duality.

Proof. Let U be Scott open in S, and let $x \in U$. Now we know

(i) $x = \sup d_j f_j x$

(cf. Compendium p.209, 3.4). Since U is Scott open and the family $d_j f_j x$ is directed, there is a j with $d_j f_j x \in U$. But S_j is continuous, whence $f_j x = \sup \Downarrow f_j x$, and so

(ii) $d_j f_j x = \sup d_j \Downarrow f_j x$.

Once more since U is Scott open, there is an element $u_j \ll f_j x$ such that

(iii) $du_j \in U$.

Let V_j be an open filter in S_j with $V_j \subseteq \hat{\uparrow} u_j$ and $f_j x \in V_j$.

Now we have $x \in f_j^{-1} V_j = \uparrow d_j V_j \subseteq d_j \uparrow u_j \subseteq U$.

This proves $\sigma(\lim S_j) \subseteq O(\lim S_j)$. The other inclusion is always true. □

The formalism of Compendium p.209, 3.4 ff. applies and allows the following Corollary, since it guarantees that a projective limit of an inverse system in which all bonding maps have lower adjoints has limit maps with lower adjoints:

6.8. COROLLARY. Let S be a projective limit in $\underline{L}$ of an inverse system of continuous semilattices in which all bonding maps have lower adjoints. Then S has duality.

We give a further sufficient condition for duality:

6.9. PROPOSITION. Let S be a projective limit of an inverse system of continuous semilattices S_j and suppose that the following condition is satisfied for the limit maps $f_j: S \longrightarrow S_j$:

(*) For each open filter U of S and each $u \in U$ there is an index j and an open filter V_j in S_j such that $f_j u \in V_j$ and $f_j^{-1} V_j \subseteq U$.

Proof. We apply Corollary 6.6: Hypothesis (*) says precisely that $\varphi(\lim S_j) \subseteq O(\lim \Sigma S_j)$. The other inclusion is always true in the situation that $\varphi(S_j) = \sigma(S_j)$.

§7. Application to general topology.

7.1. DEFINITION. A space X is called a k-space iff it is sober and $O(X)$ has duality in $\underline{U}$, i.e. $\varepsilon_{O(X)}: O(X) \longrightarrow O(X)^{**}$ is an isomorphism.

7.2. DEFINITION. For a space X we denote with $Q(X)$ the semilattice (w.r.t. finite sups (i.e. unions)) of all quasicompact saturated sets.

We recall that $Q(X)$ is an up-complete semilattice, i.e. an object of $\underline{U}$. Moreover:

7.3. FACT. / In a sober space X, the function $K \mapsto \mathcal{U}(K): Q(X) \longrightarrow \mathrm{OFilt}(X)$ which associates with a quasicompact set the filter of its open neighborhoods is well-defined and an isomorphism in $\underline{U}$. In any space X, a subset K is quasicompact iff its neighborhoodfilter of open sets $\mathcal{U}(K)$ is Scott open, i.e. is an element of OFilt X. For $U \in O(X)$ the set $\mathcal{J}(U) = \{K \in Q(X): K \subseteq U\}$ is an open filter in $Q(X)$ (w.r.t. the opposite order!). The map $U \mapsto \mathcal{J}(K)$: $O(X) \longrightarrow \mathrm{OFilt}\ Q(X)$ is an embedding. In this sense, $Q(X)$ is the dual of $O(X)$, i.e. $O(X)^* \cong Q(X)$, and the map : $O(X) \longrightarrow \mathrm{OFilt}\ Q(X) \cong Q(X)^* \cong O(X)^{**}$ is $\varepsilon_{O(X)}$.

7.4. PROPOSITION. For a sober space X, the following conditions are equivalent:

(1) X is a k-space.

(2) The function $U \mapsto \{K \in Q(X): K \subseteq U\}: O(X) \longrightarrow \mathrm{OFilt}\ Q(X)$ is surjective.

(3) " " - : - $\longrightarrow$ - is bijective.

(4) If $\mathcal{J}$ is an open filter on $Q(X)$, then $\mathcal{J}$ is open, and for all $K \in Q(X)$, the relation $K \subseteq \bigcup \mathcal{J}$ implies $K \in \mathcal{J}$.

Proof. Since the function in (2) is always injective, (2)$\Leftrightarrow$(3). The equivalence of (2) and (4) is clear from the definitions. In view of 7.3, (1) and (3) are equivalent.

This proposition points out that for a k-space, the semilattice $Q(X)$ of quasicompact saturated sets completely determines $O(X)$, and since X is sober, $O(X)$ completely determines the space X. The question, however, is whether our concept of a k-space is compatible with the classical one for separated spaces.

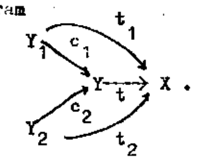
7.5. DEFINITION. Let X be a space. A test map (for X) is a proper map $t: D \longrightarrow X$ (proper: continuous and $t^{-1}K$ is quasicompact for all $K \in Q(X)$) with a locally quasicompact sober domain D. We say that a subset C of X is covered by a test if there is a test map t with $C \subseteq \mathrm{im}\ t$.

7.6. DEFINITION. We say that a subset B of $Q(X)$ is a basis of $Q(X)$ iff for each $K \in Q(X)$ there are sets $K_1, \ldots, K_n \in B$ with $K \subseteq K_1 \cup \ldots \cup K_n$.

7.7. THEOREM. Let X be a sober space on which $Q(X)$ has a basis of subsets of which are covered by test maps and whose topology is final for all test maps. Then X is a k-space.

Proof. Claim i): All $K \in Q(X)$ are covered by test maps. Proof. Let $K \in Q($ Then $K \subseteq K_1 \cup \ldots \cup K_n$ with $K_j \in Q(X)$ covered by a test map $t_j: D_j \longrightarrow X$. We let D be the coproduct of the D_j (i.e. the disjoint sum space) and $t: D \longrightarrow X$ the unique map with $t_j = t \circ \mathrm{copr}_j: D_j \longrightarrow X$. Then D is locally quasicompact and sober, and t covers K.

Claim ii): The cone of all test maps $t: Y \longrightarrow X$ with vertex X is directed and is a colimit cone. Proof. Let $t_i: Y_i \longrightarrow X$, $i=1,2$ be test maps. If Y is the coproduct of Y_1 and Y_2 and $t: Y \longrightarrow X$ the unique map induced by the t_1 and t_2, and if $c_i: Y_i \longrightarrow Y$ are the coprojections then we have a commutative diagram



This shows that the cone is directed. The hypothesis that the topology of X is final for this cone is equivalent to the assertion that this cone is a colimit cone.

Claim iii): The cocone $O(t): O(X) \longrightarrow O(Y)$ is a projective limit cone over an inversely directed basis in $\underline{\mathrm{clla}}$. Proof. The functors $O: \underline{\mathrm{SOB}} \longrightarrow \underline{\mathrm{clla}}^{op}$ and $\mathrm{Spec}: \underline{\mathrm{clla}} \longrightarrow \underline{\mathrm{SOB}}^{op}$ are adjoint on the right. Hence O transforms colimits into limits.

Claim iv) : The inverse system in Claim iii) satisfies condition (*) of 6.9. For a proof we establish the following Lemma:

7.8. LEMMA. Let $t: Y \longrightarrow X$ be a proper map between sober spaces with Y

Let $f := O(t): O(X) \longrightarrow O(Y)$. Let $\mathcal{U}$ be an open filter on $O(X)$ whose intersection is covered by t (i.e. with $\bigcap \mathcal{U} \subseteq tY$, then for each $U \in \mathcal{U}$ there is an open filter $\mathcal{V}$ on $O(Y)$ such that $f(U) \in \mathcal{V}$ and $f^{-1}\mathcal{V} \subseteq \mathcal{U}$.

Proof. By 7.3 there is a quasicompact saturated set $Q = \bigcap \mathcal{U}$ such that $\mathcal{U}$ is the filter of open neighborhoods of Q. Since t is proper, $t^{-1}Q$ is quasic compact in Y. The its neighborhood filter $\mathcal{V}$ (of open sets) is Scott open in $O(Y)$. If $U \in \mathcal{U}$ then $Q \subseteq U$ and so $t^{-1}Q \subseteq t^{-1}U = f(U)$, and so $f(U) \in \mathcal{V}$. Furthermore, if $V \in f^{-1}\mathcal{V}$, then $f(V) \in \mathcal{V}$, i.e. $t^{-1}Q \subseteq t^{-1}V$ and thus $Q = tt^{-1}Q \subseteq V$, i.e. $V \in \mathcal{U}$. For the conclusion $Q = tt^{-1}Q$ we used $Q \subseteq tY$.

The proof of the Theorem now follows from Proposition 6.9.

There is one caveat: We have no argument that the cone $t: Y \longrightarrow X$ of which X is a direct limit has a small basis or even a (small) cofinal set. An inspection of the arguments in the proof of the main theorem 6.3 show that only universal properties were used and no use was made that the index category for the j was a small set. □

7.9. PROPOSITION. Any compactly generated weakly Hausdorff space satisfies the conditions of Theorem 7.7 and is, therefore, a k-space in the sense of 7.1.

Proof. A space X is called weakly Hausdorff and compactly generated iff all quasicompact and Hausdorff subsets of X are closed and a set U in X is open iff $U \cap K$ is open in K for all compact Hausdorff subspaces K of X. The collection of all inclusion maps $t: K \longrightarrow X$, K compact Hausdorff, is a family of test maps satisfying the conditions of 7.7. □

After Proposition 7.9, the classical k-space theory is subsumed under the k-space theory proposed in Definition 7.1.

7.10. PROPOSITION. Let X be a k-space. Then the following statements are equivalent:

(1) $Q(X)$ is a continuous semilattice.

(2) X is locally quasicompact.

Proof. (2)⇒(1) is a result of Hofmann and Mislove [5]. (1)⇒(2): If (1), then $Q(X)^*$ is a continuous semilattice. Since X is a k-space, $Q(X)^* \cong O(X)$. Thus $O(X)$ is a continuous cHa and thus X is locally quasicompact. □

7.11. PROPOSITION. Let X be a sober space with a countable basis for its topology. Then the following statements are equivalent:

(1) $O(X)$ has duality.

(2) X is a k-space.

(3) $\Phi O(X)$ is sober.

Proof. By Definition 7.1 we have (1) = (2). If X has a countable basis for its topology, then every Scott open filter of $O(X)$ has a countable basis in the sense of 4.15; this follows readily from the fact that any Scott open filter on $O(X)$ is the neighborhood filter of a quasicompact saturated set. By 4.15, every Scott open filter has the refinement property, and thus, by 4.14 and 4.12, the equivalent conditions of 5.8, 5.9 and 5.10 are satisfied with $S = O(X)$. By Theorem 5.6 we then conclude that (1) and (3) are equivalent. □

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