

SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME: John Isbell	Date	M	D	Y
		11	22	81
TOPIC: Sober quotients				

It is not easy to guarantee that a  $T_0$  quotient space of a sober space is sober. For instance, if the quotient map  $X \rightarrow Y$  is two-to-one, that is not enough. (Construct  $\omega$ , with open set = upper set, by sticking together adjacent  $2^1$ 's. If you want it exactly two-to-one, add a 1.)

LEMMA. If  $S \subset X$ , if  $X$  is sober, and if the quotient space  $Y$  of  $X$  in which  $S$  is pinched to a point is  $T_0$ , then  $Y$  is sober.

Proof. Let  $f: X \rightarrow Y$  be the quotient map. If  $C \subset Y$  is irreducible closed,  $f^{-1}(C)$  is closed; and if it is irreducible, it has a dense point which gives a dense point of  $C$ . So suppose  $f^{-1}(C)$  reducible, having two disjoint, non-empty relatively open sets. It can't have two such sets that are  $f$ -saturated, for their images would be disjoint relatively open. (This depends on  $C$  being closed, so  $f|_{f^{-1}(C)}$  is a quotient map.) In particular,  $S \in C$ ,  $S \subset f^{-1}(C)$ . If  $f^{-1}(C) \subset S$ , then  $C$  has a dense point, viz.,  $S$ .

If not, we have a non-empty relatively open  $W = f^{-1}(C) \setminus S$ . There are not two disjoint relatively open sets meeting  $W$ , since subsets of  $W$  are  $f$ -saturated. Hence  $W$  is irreducible closed and has a dense point  $w$ . Now  $W$  meets  $S$ ; otherwise  $W$ ,  $S$  would be  $f$ -saturated closed proper subsets covering  $f^{-1}(C)$ , and  $f(W)$ ,  $f(S)$  would reduce  $C$ . Then in  $C$ ,  $\{f(w)\}^-$  contains  $f(W)$ ,  $S$ ,  $S$ ; This is all of  $C$ .