MORE NOTES ON CHAINS in CL-OBJECTS (from JHC) 5/28/76

REFERENCE: Notes on chains in CL-Objects (from KHH) 4/19/76

First an observation or two on the reference. On page 1, Proposition 4, it is claimed that if C is a maximal strict chain in a CL -object, then max $C \not\ll 1$. This is false. What is true is (iv)' max $C \ll 1$ iff $1 \ll 1$.

Ron Wilson observes that, using (2) on page 3 of the reference, we find that if C is a maximal strict chain in S, $\forall c \in C$, $\psi(c) = \sup_C \{d \in C : \varphi(d) \leq c\}$ and, as $\varphi(c) \leq c$, we have

$$\text{A.} \quad \forall c \in C, \psi(c) \geq c .$$

Now suppose there exists $d > c$ such that $\varphi(d) \leq c$. Then $c \ll d$ (since C is strict) and so $\varphi(d) = c$ (by definition of φ). But this yields that $[c,d]_C$ is a gap in C which implies (by Prop. 4(ii) on page 1 of the reference) that $d \in K(S)$, which in turn implies $d \ll d$, a contradiction ($\varphi(d) = c < d$). Hence, $\{d \in C : \varphi(d) \leq c\} \subseteq (\downarrow c)$ and so we also have

$$\text{B.} \quad \forall c \in C, \psi(c) \leq c .$$

Thus we have shown:

<u>Proposition</u> 1. $\forall c \in C, \psi(c) = c$.

Now, using condition (3) on page 3 of the reference and the preceding Proposition 1, we have:

<u>Proposition</u> 2. $\forall d \in S \ni \varphi(c) \leq d \leq c$, $\psi(d) = \psi(c) = c$.

Proof. $c = \psi\varphi(c) \leq \psi(d) \leq \psi(c) = c$.

Refer now to Lemma 6 on page 3 and consider $S = C = [0,1/3] \cup [2/3,1]$ as a subsemilattice of I_m. In this example, $c = 2/3$ is isolated from below in the induced topology and yet $c = \varphi(c)$ (as $c \in K(S)$). Hence, (iii) should be restated as follows (easy to prove):

(iii)' Either $c \in K(S)$ or c is not isolated from below in C in the induced topology.

The "i.e." statement should probably be isolated (set apart, that is) and labeled (iv). It is true that (i), (ii), (iii)', and (iv) are equivalent if this is done. Note also that there is a misprint in the statement of part (ii). It should be a ψ, not a φ.

Along these lines, the following is also true:

Proposition 3. If C is a maximal strict chain and $c \in C$, then T.A.E.

(1) $c \in K(S)$.

(2) $\varphi(c) \ll c$.

Moreover, if these conditions hold, then $\varphi(c) = c$.

No comments yet on accessibility other than to suggest that the definition of $s \in S$ being accessible be changed to "there exists a strict chain $C \subsetneq S \setminus \{s\} \ni s = \sup_S C$".

As to the question of a canonical way to find in a compact chain T a subchain which is strict in S (Note: not strict in T as in the reference - every chain is strict in itself) we offer the following approach.

<u>Time Out!</u> We are trying to polish off a couple of "missing links" and it is taking longer than expected. I will, at this point, send you the previous information together with some remarks of a general nature (to follow). If we do not have the entire "canonical maximal strict chain" problem solved by Friday (6/4/76), I will send you what we have and perhaps you can supply the missing links.

Let S be a complete lattice. Here are two relations on S :

(a) $x \ll y$ (x is <u>way below</u> y) if $y = \sup Y$ implies $x \leq \sup F$ for some finite set $F \subseteq Y$.

(b) $x \lll y$ (x is <u>way way below</u> y) if $y \leq \sup Y$ implies $x \leq \sup F$ for some finite set $F \subseteq Y$.

Now let S be a topological semilattice which is algebraically a complete lattice. Here is a relation on S.

(c) $x \prec y$ if $y \in (\uparrow x)^{\circ}$.

For convenience we will use $\mathfrak{L}$ to denote the category of complete lattices (with the discrete topology), $\mathfrak{C}$ to denote the category of compact semilattices with identity, and $\mathfrak{CL}$ to denote the category of compact Lawson semilattices. Here are some (easily proved) observations about the above-mentioned relations in these categories:

	$\mathcal{S}$	$\mathcal{C}$	$\mathcal{CL}$
$x \ll y \Rightarrow x \lll y$	No[1]	Yes	Yes
$x \lll y \Rightarrow x \ll y$	Yes	Yes	Yes
$x \ll y \Rightarrow x \prec y$	-	?	Yes
$x \prec y \Rightarrow x \ll y$	-	Yes	Yes
$x \lll y \Rightarrow x \prec y$	-	?	Yes
$x \prec y \Rightarrow x \lll y$	-	Yes	Yes
$x \ll y \leq z \Rightarrow x \ll z$	No[1]	Yes	Yes
$x \leq y \ll z \Rightarrow x \ll z$	Yes	Yes	Yes

[1] Consider the example of Keimel/Scott.

Here, $x \ll x$ but $x \not< \not< \not< x$, and $x \ll x \leq 1$ but $x \not< \not< 1$.

The last column of course yields that in $\mathcal{CL}$, (a), (b), and (c) are equivalent. We would very much like for someone to remove one or both of the question marks from the table.

Consider the following conditions:

(1) $x \ll y$.

(2) $y = \sup T$, with T sup-closed (algebraically) implies $x \leq t$ for some $t \in T$.

(3) $y = \sup D$, with D up-directed implies $x \leq d$ for some $d \in D$.

(1)' $x \lll y$.

(2)' $y \leq \sup T$, with T sup-closed implies $x \leq t$ for some $t \in T$.

(3)' $y \leq \sup D$, with D up-directed implies $x \leq d$ for some $d \in D$.

We have found it convenient to use on occasion that (1), (2), and (3) are equivalent and that (1)', (2)', and (3)' are equivalent (in $\mathcal{S}$). Also, along these lines (referring back to ATLAS), consider the following conditions:

1. x is a compact element.
2. $x \leq \sup X$ implies $x \leq \sup F$ for some finite set $F \subseteq X$.
3. $x \leq \sup T$, with T sup-closed implies $x \leq t$ for some $t \in T$.
4. $x \leq \sup D$, with D up-directed implies $x \leq t$ for some $t \in T$.
5. $x \lll x$.
6. $x \ll x$.
7. $\uparrow x$ is open.

In $\mathcal{S}$ 1 - 5 are equivalent. In $\mathcal{C}$ 1 - 6 are equivalent. In $\mathcal{CL}$ 1 - 7 are equivalent. In $\mathcal{C}$ 7 implies the equivalent conditions 1 - 6 . <u>Question</u>: Does 1 imply 7 in $\mathcal{C}$?

Let S be an $\mathcal{S}$-object. For $x \in S$ let us denote $\{y : y \ll x\}$ by $\twoheaddownarrow x$, $\{y : x \ll y\}$ by $\twoheaduparrow x$, $\{y : y \lll x\}$ by $\twoheaddownarrow\!\!\twoheaddownarrow x$, and $\{y : x \lll y\}$ by $\twoheaduparrow\!\!\twoheaduparrow x$. If S is a $\mathcal{C}$-object and $x \in S$, then we will denote $\{y : x \in (\uparrow y)^o\}$ by $(\downarrow x)_o$ (as in the Notes on Stralka's Amalgamation Theorem of Nov.-Dec. 1975). The following observations seem to be "in the spirit" of the recent mailings:

Proposition 4.

1. $(\twoheaduparrow x) \cap (\twoheaduparrow y) = \twoheaduparrow(x \vee y)$.

2. $(\threeheaduparrow x) \cap (\threeheaduparrow y) = \threeheaduparrow(x \vee y)$.

3. $(\uparrow x)^{o} \cap (\uparrow y)^{o} = \uparrow(x \vee y)^{o}$.

4. $\twoheaddownarrow x$ is a lattice ideal with the property that if L is any lattice ideal such that $x = \sup L$, then $\twoheaddownarrow x \subseteq L$.

5. $\threeheaddownarrow x$ is a lattice ideal with the property that if L is any lattice ideal such that $x \leq \sup L$, then $\threeheaddownarrow x \subseteq L$.

6. $(\downarrow x)_o$ is a lattice ideal with the property that if L is any lattice ideal such that $x \leq \sup L$, then $(\downarrow x)_o \subseteq L$, provided S is compact. To see that the conclusion fails in the non-compact setting let $x = 1$ in the Keimel/Scott example.

One might rightfully ask "Why all the fuss about distinguishing between $\ll$, $\lll$, and $\prec$?" since they are all equivalent in $\mathcal{CL}$. What we are keeping in the backs of our minds is a hoped for duality theory for $\mathcal{C}$.

Now let us say that an $\mathcal{S}$-object S satisfies property:

"1" if for each $x \in S$ we have $x = \sup (\downarrow x)_o$ (provided S is topological)

"way 1" if for each $x \in S$ we have $x = \sup \twoheaddownarrow x$.

"way way 1" if for each $x \in S$ we have $x = \sup \threeheaddownarrow x$.

Corollary to Proposition 4.

1. If S satisfies property "1" (and is compact), then $(\downarrow x)_o$ is the unique smallest lattice ideal whose sup is $\geq x$.

2. If S satisfies property "way 1" , then $\twoheaddownarrow x$ is the unique

smallest lattice ideal whose sup is x .

3. If S satisfies property "way way 1" , then $\downarrow\!\!\downarrow\!\!\downarrow x$ is the unique smallest lattice ideal whose sup is $\geqslant x$.

Note: The Keimel/Scott example shows that S can satisfy property "way 1" without satisfying property "way way 1" . It is trivial to show that if S is a $\mathcal{C}$-object then S satisfies property "1" iff S is Lawson.

Perhaps some of these observations will prove useful to others at least to the extent of giving alternate "ways to think" (please forgive me for using the word "way" yet another time!)

More to come . . .

The Carruth Seminar
J. H. Carruth
C. E. Clark
J. W. Crawley
E. L. Evans
R. L. Wilson