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TOPIC A continuous poset whose compactification is not a continuous poset. The square is the injective hull of a discontinuous CL-compact poset.

REFERENCES

[Reh_1] Hoffmann, Rudolf-Eberhard, The CL -compactification and the injective hull of a continuous poset, Amer.J.Math. 1983 .

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[Khh] Hofmann, Karl Heinrich, The category DC of completely distributive lattices and their free objectsSCS Memo 11-24-81

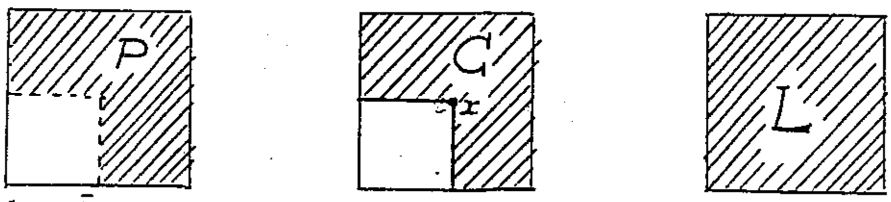
[HM] Hofmann,K.H. and M.W.Mislove, Universal constructions for completely distributive lattices, nearly finished.

We propose a simple example of a continuous poset which may help to illustrate a number of phenomena of a slightly delicate nature as they arise in the context of the injective hull and the CL-compactification.

EXAMPLE. We let L be the square $[0,1]^2$ with its usual order structure, and we define subsets $P \subset C \subset L$ as follows:

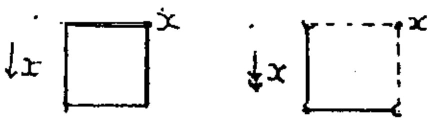
i) $C = L \setminus$ hole , where hole = $\{(x,y):\ 0 < x,y < 1/2\}$.

ii) $P = L \setminus$ HOLE , where HOLE = $\{(x,y):\ 0 < x,y \leq 1/2\}$.



Clearly, $\bar{P} = C$, where the closure is taken in the natural,i.e. the CL-topology. We go through the various properties of the examples before us.

1. P is a continuous poset. Its CL-topology is locally connected and connected.
   The proof proceeds by inspection. The way-below relation is that induced from L. The CL-topology is that induced from the CL-topology of L. Note that the CL-topology is locally compact in all but two points.
2. C is not a continuous poset.
   Proof. We consider the point $x = (1/2,1/2)$. We observe $\downarrow x = \{y \in C:\ pr_1 y,\ pr_2 y \leq 1/2\}$, and $⇓ x = \{y \in \downarrow x:\ (pr_1\ y = 0$ and $pr_2\ y < 1/2)$ or $(pr_2 y = 0$ and $pr_1\ y < 1/2)\}$. Then $x = \sup ⇓ x$ (which is predicted by the general theory to which we will come back later), BUT $⇓ x$ is not directed.



3. L is a continuous (and indeed a completely distributive) lattice. (What else is new?)

4. L <u>is the topology-induced injective hull of</u> P <u>and of</u> C.

Proof. We have to recall some background at this stage. There are several ways to understand this claim. Firstly, one may refer to a recent characterization theorem due to Reh ([Reh_1], p.15, TEHOREM 1.7):

<u>THEOREM</u>. Suppose that a continuous poset P is contained in a complete lattice L and that the order of P is the induced one. Then L is the injective hull of P iff

(i) P is sup-dense in L (i.e. P order cogenerates L)

(ii) P is closed in L under the formation of directed sups.

(iii) The way below relation of P is induced by the way below relation of L.

(iv) The <u>CL</u>-subalgebra generated in L by P is L.

(In lieu of (iii) one may also write (cf.loc. cit. Lemma 1.1 on p. 11):

(iii') If $x \ll_P y$ in P then $x \ll_L y$.)

The conditions (i)-(iv) are satisfied for P in L. Bb Lemma 2,p.235 applies to $P \subseteq C \subseteq L$. This proves claim 4.

It is instructive to consider P as embedded into the completely distributive lattice $\Upsilon(P)$ of all Scott closed sets under $x \mapsto \downarrow x$. Then $\lambda(P)$, the topology-induced injective hull of Reh is concretely realized in $\Upsilon(X)$ from the image S of P (which is the cospectrum of $\Upsilon(X)$) as follows:

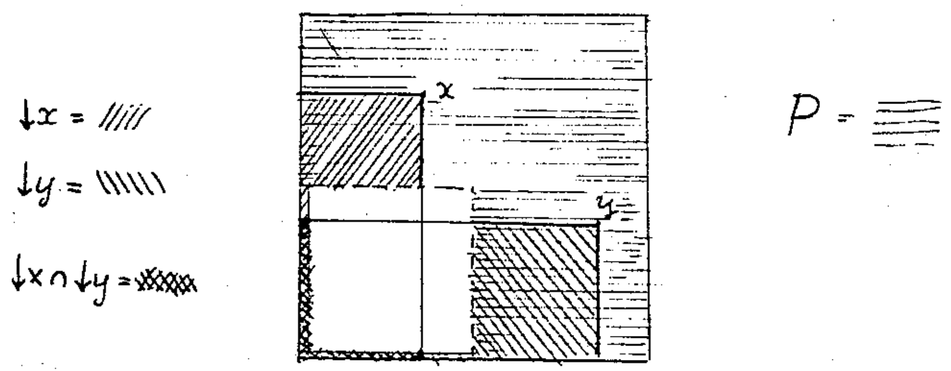
(a) First form the smallest $\wedge$-subsemilattice containing S.

(b) Add all filtered infs and 1.

(c) Add all directed sups.

(See Reh_1, 5.4,p. 32, as well as Khh and HM.)

It is clear in our example that step (a) already fills the <u>HOLE</u> in P and thus yields L, which is already a continuous lattice, hence will not be enlarged by steps (b) and (c).



5. C is the CL -compactification of P.

Proof. According to the definition, due to Reh [Reh_1 ,p.20, Section 2.3] the CL-compactification of a continuous poset P is its closure in the inject. L of P with respect to the CL-topology. In our example, the CL-topology of L is the natural topology of the square. It is then clear that C is the CL-compactification of P .

6. P is a continuous poset (with locally connected CL-topology) whose CL -compactification is not a continuous poset.

It should be clear that, in the construction of $L = [0,1]^2$ the unit interval $[0,1]$ may be replaced by a suitable totally ordered algebraic lattice as long as the element replacing 1/2 in $[0,1]$ is neither isolated from below nor from above. We can therefore state

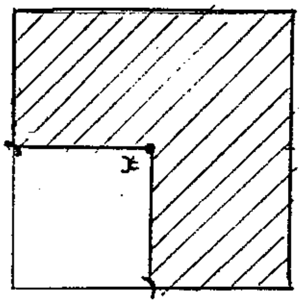
7. There exist algebraic posets whose CL-compactifications are not algebraic posets (and not even continuous posets).

8. The space C with its Scott topology is a T_0-space with an injective hull. which is not a continuous poset (relative to the specialisation order)

There are simpler examples for this phenomenon. The boundary of the square will do.

9. The space C with its Scott topology is a T_0-space with an injective hull and C contains a point x (viz, x = (1/2,1/2)) with a neighborhood W such that for no point $w \in W$ the point x is in the interior of $\uparrow w$.
(In the terminology of Bb pp.239 and 240 there is no open neighborhood V of x such that $W \cap \Gamma_0 V \neq \emptyset$.)

Proof. Indeed the set $W = C \setminus (\downarrow x)^-$ (= $\{(u,v) \in C$: (if u= 0 then v > 1/2) and (if v = 0 then u > 1/2)$\}$) is such a neighborhood.



We summarize: For a sober space X let λX be the essential hull in the sense of Bb. Consider the two statements:

(1) λX is injective (i.e. X has an injective hull)

(2) X is a continuous poset in its specialisation order and its topology is the Scott topology for this poset structure.

Then we have the following facts:

FACT i. (2) implies (1)

FACT ii. (1) does not imply (2).

We also consider the following statements:

(I) X has an injective hull. ($\Leftrightarrow$ (1) above).

(II) For each point x in X and each open neighborhood W of x there is an open neighborhood V of x such that for some $w \in W$ the relation $V \subseteq \uparrow w$ holds.

The we have the following facts:

FACT i. (II) implies (I)

FACT ii. (I) does not imply (II).

Concerning this summary, cf. [Reh_2], p.150 Theorem 3.4 and p.154, Theorem 3.14, and [Bb] p.240 Corollary 2.

What are the positive results which we have in the context of our discussion? Firstly, we owe to Reh the following Theorem which is not hard to prove once one has the general characterisation theory for injective hulls of continuous posets developed in the first chapter of Reh_1

THEOREM. If S is a continuous semilattice, thenthe CL-compactification C is a continuous lattice and agrees with the topology-induced injective hull L of S. [Reh_1],p.24, Theorem 3.3 .

Secondly,
we repeat the necessity portion of Reh's characterisation theorem [Reh_1] p.15,1.7:

COROLLARY. If a continuous lattice L contains a continuous poset P such that L is the injective hull of P, then P is closed under directed sups and inherits the way below relation and L is generated by P in each of two ways: L is the set of all L-sups of P,and L is the set of directed sups in L of the set S of all L-infs of P.

This corollary controls quite sharply the continuous posets, of which L can be the injective hull. Let us illustrate this in the following result:

THEOREM A. Let L be a continuous lattice and $P \subseteq L$ a continuous poset such that L is the injective hull of P. Then we have the following conclusions:

(i) The CL-compactification $\overline{P}$ in L is inf-dense in L (i.e. $\overline{P}$ order generates L, i.e. every element of L is an inf of elements in $\overline{P}$).

(ii) $\text{IRR } L\setminus\{1\} \subseteq \overline{P}$.

(iii) If L happens to be completely distributive, then $\overline{P}$ contains both Spec L and Cospec L.

(iv) For each point $x \in \overline{P}$ we have $x = \sup \Downarrow x$ (in $\overline{P}$).

Proof. By Reh's characterisation theorem, P topologically generates L in the sense of the Compendium p. 243, Definition 2.3. Then (i) and (ii) follow from Compendium p.244 , Proposition 2.4. In order to prove (iii), assume that L is completely distributive. Then the Lawson topologies of L and L^{op} agree (Compendium p.318). By Reh's characterisation theorem, P is order generating in L^{op}; then Compendium p.243, Theorem 2.1 applies to L^{op} and shows that Cospec L is in $\bar{P}$. Concerning (iv) we recall that $x = \sup \downarrow_L x, x' = \sup (\downarrow x' \cap P)$ for $x' \ll x$. We conclude $x = \sup (\downarrow_L x \cap P)$, whence (iv). □

Remarks. i) Recall that 0 and 1 may or may not be in $\bar{P}$. ii) From (i) we recover once more that L is the MacNeille completion of $\bar{P}$ (see [Reh$_1$], 5.6 on p.39).

We illustrate . A in the following example:

EXAMPLE. Let P be a continuous poset in the square $L = [0,1]^2$ such that L is the injective hull of P. Then $\bar{P}$ contains the boundary of the square L.

Proof. We have Spec L ∪ Cospec L ∪ {0,1} = boundary L . Apply the preceding Corollary.

Remark. In this particular instance, one notices without difficulty directly that P itself must contain the lower half of the boundary (except, possibly, for 0) (i.e., Cospec L $\subseteq$ P).

CONSEQUENCE. If P is a continuous poset in the square $L = [0,1]^2$ such that L is the topology-induced injective hull of P, then P is not contained in the boundary B of the square.

Proof. If P were contained in the boundary, then $\bar{P}$ would be contained in it. The preceding result then shows that $\bar{P}$ would be equal to the boundary, and since P is up-complete, we conclude $B \setminus \{(0,0)\} \subseteq P$, and thus P is not a continuous poset.

We will prove below the following

CLAIM. The square $L = [0,1]^2$ is the essential hull of its boundary (with the Scott topology).

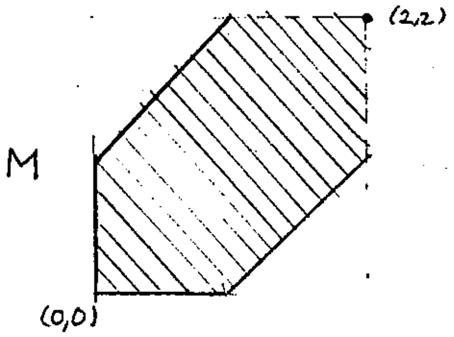
This example is of interest for the following reason: Since there is no obvious necessary condition on a sober space X to have an injective hull one might be tempted to conjecture that a sober space X with an injective hull might at least have to be an essential extension of a continuous poset (in its Scott topology). However, the CLAIM shows that this is false, too. This follows from the CONSEQUENCE above.

Now we establish the CLAIM. We have very little in the line of general theory which would allow us to conclude this right away; so we have to establish the Claim from scratch. But that process also is a good exercise and illustration of Bb's and Reh's theory of essential hulls.

We let L be the square $[0,1]^2$ and B the boundary.

LEMMA . Let M be the poset in the plane consisting of all (x,y) with $0 \leq x,y < 2$ and $-1 \leq x - y \leq 1$ or $x=y=2$. Let M^0 be M with an isolated 0 attached. Then the lattice $\Gamma(B)$ of Scott closed sets of the boundary is isomorphic to M^0.

Proof



We define $f: M^{o} \longrightarrow \mathcal{T}(B)$ as follows:

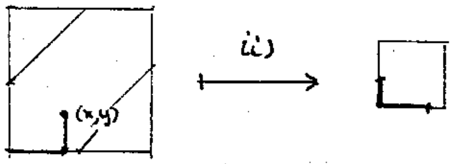
i) If 0 is the isolated zero of M^{o}, then $f(0) = \emptyset$. Also $f(2,2) = B$.

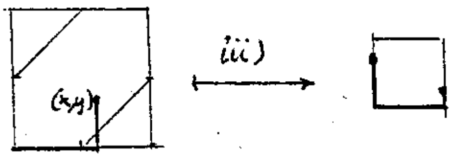
ii) If $0 \leqslant x,y \leqslant 1$, then $f(x,y) = {\downarrow}_B(x,0) \cup {\downarrow}_B(0,y)$.

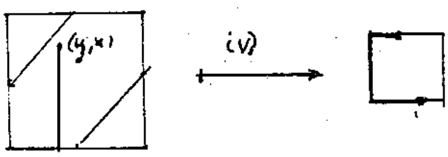
iii) Suppose that $0 \leqslant x \leqslant y \leqslant 1$ and that $y = 1$ implies $x < 1$. Then
$f(1+x,y) = {\downarrow}_B(1,x) \cup {\downarrow}_B(0,y)$.

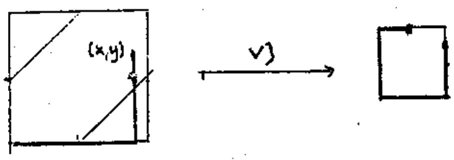
iv) Suppose that the same conditions hold as in iii). Then
$f(y,1+x) = {\downarrow}_B(x,1) \cup {\downarrow}_B(y,0)$

v) If $0 \leqslant x,y < 1$, then $f(1+x,1+y) = {\downarrow}_B(x,1) \cup {\downarrow}_B(1,y)$.









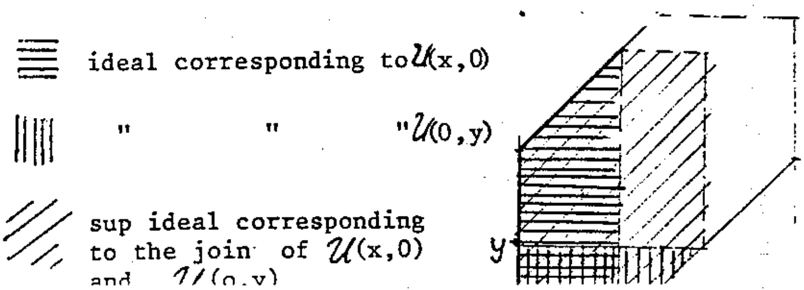
Then f is the required isomorphism.

The function $b \mapsto f^{-1}({\downarrow}b): B \longrightarrow M^{o}$ maps B onto the boundary of M under the preservation of order and of all existing infs.

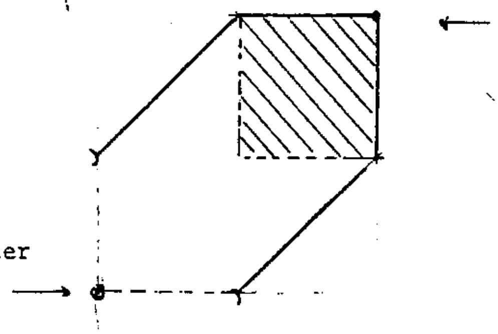
The filters of open sets on B are the ideals of $\mathcal{T}(B)$, hence of M^{o} (up to natural identification). The neighborhood filter of $(x,0)$ in B corresponds to the ideal $\{(u,v) \in M : u < x\} \cup \{0\}$, and the neighborhood filter of $(1,x)$ in B corresponds to the ideal

$$\{(u,v) \in M : u < 1+x\} \cup \{0\}$$

There are two corresponding cases for the points on the other hemisphere of the boundary. The join filters in the sense of Bb correspond to the joins of any two of the four types of ideals. We sketch one:



Each of the occurring join ideals is uniquely characterized by its sup in M. We see that the space of sup-ideals is represented by the following subset of M, in the induced order:

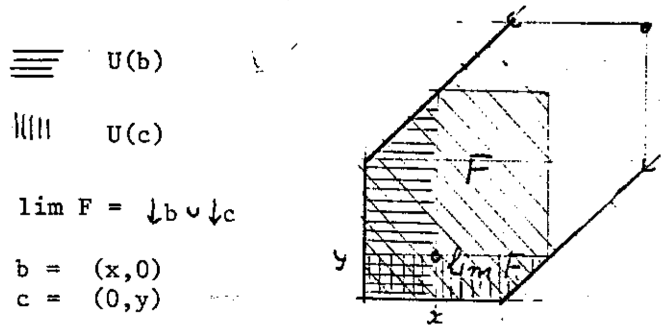


neighborhood filter of (1,1)

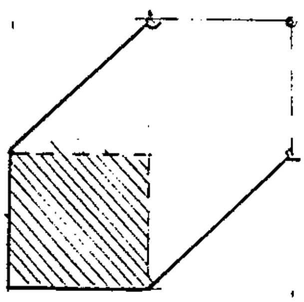
(1)

neighborhood filter of (0,0)

As Reh has pointed out, a join filter of open sets may be represented by an element in $\Upsilon(B)$, namely, its set of limit points. A point $b \in B$ is a limit point of a filter F of open sets iff $U(b) \in F$ (where U(b) is the filter of open neighborhoods of b). We denote with lim F the set of limit points of F. Example:



A precise inspection of the situation shows that the space of limit sets of all join filters in $\Upsilon(B)$ is



(2)

The posets in both (1) and (2) are both order isomorphic to the square- which is what we had to show.

It is interesting to observe how the square is embedded into $M^o = \Gamma(B)$ as the injective hull $L=\lambda(B)$ of B. It is closed under infs, but not under directed sups. Thus it is not embedded as a CL -subalgebra. Moreover, the embedding is such that $x \ll_L y$ implies $x \ll_M y$ but if $x < y$ holds on one of the feelers, then $x \ll_M y$, but not $x \ll_L y$. (This phenomenon is only possible since L is not closed under directed sups in M - see [Reh_1], p.12, 1.2.b.) . The Scott topology of $\Gamma(B)$ does not induce the Scott topology of the Reh-embedded essential hull L, but the Scott topology does induce the Bb-embedded essential hull L (see (2), resp. (1)). This phenomenon illustrates the difficulties one has in explicitly describing the Reh-embedded essential hull L in the lattice of Scott closed sets. This is the reason why Reh had to "transport" the Scott topology from the Bb-embedded essential hull rather than to say explicitly which of the standard topologies on the lattice of closed sets might induce the essential hull topology.

These remarks are also relevant to Reh's approach to the Fell compactification of a continuous poset- which is none other than the CL-compactification (see Reh_1 pp.35 ff.) For a continuous poset P, Reh considers the lattice $\Gamma(P)$ of Scott closed sets (in Reh called "A(X)") and shows that the injective hull L of P is both Scott- and CL-embedded in $\Gamma(P)$. The example of the boundary B of the square L above shows that this fails in the absence of continuity of P; the reason is that $\Gamma(P)$ is continuous iff (as a complete lattice order cogenerated by its cospectrum) $\Gamma(P)$ is completely distributive iff P is continuous .(We assume here that P is at any rate sober in its Scott topology which we know is not necessarily the case even for complete lattices (Johnston-Isbell).) How the continuity of $\Gamma(P)$ is utilized was shown in Khh SCS 11-24-81 and can be found in Reh_1 in various places, such as e.g. p.36,37;5.3. Thus, while the CL-compactification of a continuous poset may be readily formed within $\Gamma(P)$ as a CL-closure this may be doubtful in the general case, although the examples in this memo do not clearly yield counterexamples for this possibility.

There is almost no limit to one's imagination in playing with the square. Here are some variations to the theme discussed in this memo: