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(The essential hull revisited)

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0. Introduction.

Banascheski's paper on essential hulls is, next to Scott's seminal article on continuous lattices, one of the very early sources of their theory (B); since it introduces features relating to continuous posets which are at the focus of more current research long before the concept of a continuous poset congeled it has, in recent years, spawned a whole line of research (A).

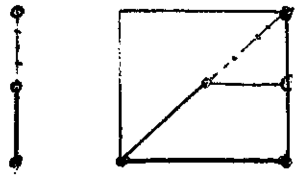
The objective of this memo is a review of Banaschewski's paper and its principal results. The review may be timely for the following reasons: (i) We now have a well developed theory of continuous lattices and standard sources (C). (ii) The original article (B) is not easily readable in all places, and the Corollary 2 to Proposition 3, on which many results in (A) are based, is false (D) (iii) The basic ideas of (B) are important and viable and it might be useful to take stock of what the precise status is.

The basis of our discussion here will be in the framework of continuous and algebraic lattices. At a later stage, we merge into the language of filters which Banaschewski uses to construct the essential hull of a space. R.-E.Hoffmann has given alternative constructions A2).

1. Sup-closed subsets in continuous lattices.

We fix a continuous lattice L. From the theory of Galois connections (C,p.18 ff.) we know that there is a canonical bijection between the set of sup-closed subsets A of L and the set of kernel operators $k: L \longrightarrow L$; indeed each kernel operator k determines a sup-closed subset $A = k(L)$, and from each sup-closed subset A we obtain a kernel operator k via $k(x) = \sup(\downarrow x \cap A) = \max(\downarrow x \cap A)$. The two operations invert each other. All of this requires the completeness, but not the continuity of L. Our objective is to find explicit criteria for the continuity of A if L is continuous. We know from C, p.63, Theorem 2.14 that A is continuous whenever the associated kernel operator k is continuous relative to the Scott topology (cf.C,p.112).

The converse may fail, as shows the example $k:[0,1] \longrightarrow [0,1]$ with $k(x)= x$ for $x=1$ or $x \leq 1/2$ and with $k(x)= 1/2$ else.



However, there are cases, when the continuity of k is necessary and sufficient for the continuity of A.

1.1. LEMMA. Let $k:L \longrightarrow L$ be a kernel operator on a continuous lattice and $A = k(L)$. Suppose that L is the smallest subset of L containing A and being closed under arbitrary infs and directed sups (i.e. A generates L as a CL- algebra). Then the following are equivalent:

(1) k is continuous. (2) A is continuous and topologically embedded in L.

Remarks. i) Here and in the following, continuity of a map between posets always refers to the Scott topologies, unless noted otherwise. ii) A special case of Lemma 1.1 is implicit in Proposition 3 of (B).

Proof. We only have to prove (2) $\Rightarrow$ (1). Thus suppose that A is continuous. Then A is a continuous retract of L, since A is injective (with its Scott topology) in the category of T_0-spaces and is embedded into the space $\Sigma L = (L, \sigma(L))$, where $\sigma(L)$ is the Scott topology. (See C, p.121 ff.) Thus there is a continuous projection operator $p:L \longrightarrow L$ with $p(L)= A$ (cf. C, p.21, Definition 3.8.i). Next we define $L_k = \{x \in L: p(x) \leq x\}$ (see C, p.22, Lemma 3.11 and p.63, Theorem 2.14). Now L_k is closed in L under infs and directed sups (C, p.63, loc.cit.). For $a \in A$ we have $p(a)= a$, hence $a \in L_k$. Since A CL-generates L by hypothesis, we have $L_k=L$, i.e. p is a kernel operator with $p(L) = A = k(A)$. Then $x \geq k(x)$ implies $p(x) \geq pk(x) = k(x)$, and likewise $x \geq p(x)$ implies $k(x) \geq k(x)$. Thus $k=p$ and k is continuous. □

We record the following observation:

1.2. EXERCISE. If A is sup-closed in a continuous lattice L, then the following statements are equivalent:

(1) A is continuous and topologically embedded in L.

(2) For each $a \in A$ and $x \in L$ the relation $x \ll_L a$ implies the existence of a $b \in A$ with $x \leq b \ll_L a$

For all $a,b \in A$ the relation $a \ll_A b$ is always a consequence of $a \ll_L b$, but if (1) and (2) hold, then the converse is also true. □ (Cf. C, p.181, Corollary 1.7

We omit the (simple) proof and now concentrate on the continuity of a kernel operator.

1.3. LEMMA. Let k be a kernel operator on a continuous lattice. Then the following statements are equivalent:

(1) k is continuous.

(2) $k(x) = \sup k(\Downarrow x)$ for all $x \in L$.

(3) $a = \sup k(\Downarrow a)$ for all $a \in A$ $(=k(L))$.

Remark. The way below relation $\ll$ refers to L unless otherwise specified.

Proof. (1) <=>(2) :See C,p.112,Proposition 2.1, (1) <=>(5); the implication (1)=>(5) uses only the continuity of the domain S, and a simple modification of the proof of (5)=>(4) shows (5) =>(1) without the hypothesis that the range T be continuous. In other words, the equivalence of (1,2,3,5) holds whenever S is continuous and T is complete.

(2)=>(3): Trivial since $k(a)= a$ for $a \in A$.

(3)=>(2): Let $x \in L$ and set $a = k(x) \in A$. Then $a = \sup k(\quad a)$ (by (3)) $\leq \sup k(\Downarrow x)$ (since $a = k(x) \leq x$) $\leq k(x) = a$. Thus $k(x) = \sup k(\Downarrow x)$. □

Now suppose that G is an arbitrary subset of the continuous lattice L and let A be the set $\{x \in L: \ x = \sup (\downarrow x \cap G)\}$ which is sup-generated by G Let k be the kernel operator associated with A.

1.4. <u>LEMMA</u>. The following statements are equivalent:

(1) k is continuous.

(2) $a = \sup(\Downarrow a \cap G)$ for all $a \in A$.

(3) $g = \sup(\Downarrow g \cap G)$ for all $g \in G$.

Proof. (2) ⇒ (1): By 1.2 we know $\Downarrow a \cap G$ = ~~$\Downarrow_L a \cap G \subseteq \Downarrow_A a$, whence $a = \sup(\Downarrow a \cap G)$ $\leq \sup \Downarrow_A a \leq a$. Thus $a = \sup \Downarrow_A a$, i.e. A is continuous. Also~~ $\Downarrow_L a \cap G \subseteq k(\Downarrow_L a)$, whence $a = \sup (\Downarrow a \cap G) \leq \sup k(\Downarrow a) \leq a$. Thus $a = \sup k(\Downarrow a)$, whence k is continuous by 1.3.

(1) ⇒ (2): If k is continuous, then A is continuous. Let $a \in A$ and suppose $x \ll a$. We find an $x' \in L$ with $x \ll x' \ll a$ by the interpolation property (C p.46), and then we find a $b \in A$ with $x' \leq b \ll a$ by 1.2. Now $b=\sup(\Downarrow b \cap G)$ by hypothesis.Since $x \ll b$ there is a finite set $F \subseteq \Downarrow b \cap G$ sucht that $x \leq \sup F$. Since $\sup F \ll a$ we conclude $F \subseteq \Downarrow a \cap G$. Thus $x \leq \sup F \leq \sup(\Downarrow a \cap G) \leq a$. Since $x \ll a$ was arbitrary and $a = \sup \Downarrow a$ as L is continuous we conclude $\sup(\Downarrow a \cap G) = a$.

(2) ⇒ (3) is trivial.

(3) ⇒ (2): $a = \sup(\Downarrow a \cap G) = \sup \{g \in G: g \leq a\} = \sup_{g \in G, g \leq a} \sup_{h \in G, h \ll g} h$ *(by (3))*

$= \sup \{h \in G: h \ll g \leq a \text{ for some } g \in G\} \leq \sup(\Downarrow a \cap G) \leq a$, whence $a = \sup(\Downarrow a \cap G)$. □

1.5. <u>LEMMA</u>. Let L be a complete lattice and suppose that L is inf-generated by G (i.e. $x = \inf (\uparrow x \cap G)$ for all $x \in L$. Let $A = \{x \in L: \ x = \sup(\Downarrow x \cap G)\}$. Then the complete lattice A is inf-generated by G WITHIN A ,i.e. $x = \inf_A(\uparrow x \cap G)$ for all $x \in A$.

Proof. Let $k: L \longrightarrow L$ be the kernel operator associated with A. Let $g: L \longrightarrow A$ be the corestriction of k; then g is upper adjoint to the inclusion $d: A \longrightarrow L$, hence preserves infs. Now let $x \in A$. By hypothesis, $x = \inf(\uparrow x \cap G)$, whence $x = k(x)$ $= g(x) = \inf_A g(\uparrow x \cap G) = \inf_A(\uparrow x \cap G)$. □

We are ready for the principal Theorem of this section:

1.6. THEOREM. Let G be a subset of a continuous lattice L and let A denote the subset $\{x \in L: x = \sup(\downarrow x \cap G)\}$ which is sup-generated by G. Consider the following conditions:

(1) A is a continuous lattice in its own right and is embedded in L w.r.t. the Scott topologies.

(2) $g = \sup(\downarrow g \cap G)$ for all $g \in G$.

Then (2) $\Rightarrow$ (1) and if the smallest subset of L, which contains A and is closed under arbitrary infs and directed sups, is L, then both conditions are equivalent.

Proof. By 1.1, conditions (1) is equivalent to the continuity of the kernel operator k associated with A, provided that L is the smallest subset of L containing L and being closed under infs and directed sups. By 1.4, the continuity c of k is equivalent to (2). Since (1) follows from the continuity of k in general, the Theorem is proved. □

1.7. REMARK. If in addition to the hypotheses of 1.6, the set G inf-generates L, then G inf-generates A within A, i.e. $a = \inf_A(\uparrow a \cap G)$ for all $a \in A$. In this case, G is both meet and join dense in A.

Proof. Lemma 1.5 □

1.8. COROLLARY. Suppose that G is a subset of an algebraic lattice and A is defined as in 1.6. Then of the following conditions, (2) $\Rightarrow$ (1):

(1) A is a continuous lattice and is topologically embedded,

(2) $g = \sup\{h \in G: \text{there is a } c \in K(L) \text{ such that } h \leq c \leq g\}$ for all $g \in G$.

If L is the smallest subset of L containing A which is closed under infs and directed sups, then both conditions are equivalent. Remark 1.7 applies to this special case.

Proof. The equivalence of conditions 1.6(2) and 1.8(2) is an immediate consequence of C p.86, Proposition 4.5. □

Notice that in 1.6 and 1.8 the continuity of A is determined solely by the way by which the generating set G of A is embedded into L.

2. Algebraic lattices.

2.1. DEFINITION. Let L be an algebraic lattice. We say that a subset G of L is K-generating provided that the following conditions are satisfied:

(1) $G \subseteq \operatorname{Irr} L$ where Irr L (the set of completely irreducible elements of L, cf. C, p.92, Definition 4.19)

(2) For all $c \in K(L)$ we have $c = \inf(\uparrow k \cap G)$.

Notice that (2) is equivalent to

(2') If $c, c' \in K(L)$ and $c \not\leq c'$, then there is a $g \in G$ with $c \not\leq g$ and $c' \leq g$.

2.2. LEMMA. Let L be an algebraic lattice and G a K-generating subset. Then L is the smallest subset of L containing G and being closed under arbitrary infs and directed sups. In particular, the $1-{}_\wedge$inf-semilattice generated by G is CL-dense in L.
Proof. Let L' be the smallest aubset of L containing G which is closed under infs and directed sups. (Cf.C, p.60, Definition 2.6 ff.).
By condition 2.1 (2) it follows that $K(L) \subseteq L'$. Since L is algebraic, for each $x \in L$ we have $x = \sup(\downarrow x \cap K(L))$ and $\downarrow x \cap K(L)$ is directed (see C,p.85 ,4.3 and 4.4). Hence $x \in L'$. Thus $L' = L$. The remainder follows from C,p.146, Theorem 1.11. □

2.3. REMARK. For a subset G of Irr L in an algebraic lattice L, the following statements are equivalent:

(1) G = Irr L.

(2) $x = \inf(\uparrow x \cap G)$ for all $x \in L$.

Proof. C, p.93, Theorem 4.23. □

In particular, if G is a K-generating set which is properly smaller than Irr L, then for at least one $x \in L$ we must have $x \neq \inf(\uparrow x \cap G)$, and vice versa.

2.4 NOTATION. For a K-generating subset G in an algebraic lattice L we write

$$\lambda(G) = \{x \in L: x = \sup(\downarrow x \cap G)\}.$$

Then $\lambda(G)$ is a sup-closed lattice in L which is a complete lattice in its own right. □

2.5. THEOREM. Let L be an algebraic lattice and G a K-generating subset (2.1). Then the following statements are equivalent:

(1) $\lambda(G)$ is a continuous lattice and is topologically embedded w.r.t. the Scott topologies.

(2) For each $g \in G$ we have $g = \sup\{q \in G: q \leq c \leq g \text{ for some } c \in K(L)\}$.

(3) $(\forall g \in G, r \in \text{Irr } L)\ g \not\leq r = (\exists q \in G, c \in K(L))\ q \not\leq r$ and $q \leq c \leq g$.

(4) For each $g \in G$ and each $c \in K(L)$ with $c \leq g$ there are finitely many $p_1, \ldots, p_n \in G$ and a $d \in K(L)$ such that $c \leq p_1 \vee \ldots \vee p_n \leq d \leq g$.

Proof. By 1.8 we know that (1) and (2) are equivalent. (2) ⇒ (3) : Let $g \in G$ and $r \in \text{Irr } L$ with $g \not\leq r$. Since g is the sup of all $q \in G$ with $q \leq c \leq g$ for some c there must be one of these q with $q \not\leq r$. Conversely, (3) ⇒ (2):

Let $g \in G$ and set $g' = \sup \{q \in G: q \leq c \leq g$ for some $c \in K(L)\}$. Assume $g' \neq g$. Since clearly $g' \leq g$ we have $g \not\leq g'$ and thus, since Irr L is order generating , there is an $r \in$ Irr L with $g' \leq r$ and $g \not\leq r$. By (3) we find a $q \in G$ and a c K(L) such that $q \not\leq r$ and $q \leq c \leq g$. By the definition of g' this q satisfies $q \leq g' \leq r$, and this is a contradiction.

(2) $\Rightarrow$(4): Let $c \leq g$ with $c \in K(L)$ and $g \in G$. Since $c \ll c$, whence $c \ll g$, from (2) we obtain a finite sequence of elements $p_j \leq d_j \leq g$, $j = 1,\ldots,n$ with $p_j \in G$, $d_j \in K(L)$ and $c \leq p_1 \vee \ldots \vee p_n$. If we set $d = d_1 \vee \ldots \vee d_n$, then $d \in K(L)$, since K(L) is closed under finite sups, and condition (4) is satisfied.
(4) $\Rightarrow$(2): Assume (4) and take an arbitrary $g \in G$ and an arbitrary $c \in K(L)$ with $c \leq g$. Let p_j and d be as in (4). Then $c \leq p_1 \vee \ldots \vee p_n$ $\sup \{q \in G : q \leq e \leq g$ for some $e \in K(L)\} \leq g$. Since $g = \sup(\downarrow g \cap K(L))$ by the definition of and algebraic lattice (cf.C, p.85,Definition 4.4), conditions (2) follows. □

So far we considered $\lambda(G)$ as a lattice in its order structure. There are, of course, numerous possibilities of endowing $\lambda(G)$ with a topology. in the light of later applications we make the following convention:

2.6.CONVENTION. Unless specified otherwise, we consider on $\lambda(G)$ the topology induced from the Scott topology of L,i.e. the topology generated by the sets $\uparrow c \cap \lambda(G)$, $c \in K(L)$ as basic sets. In accordance with the notation in C we denote this topology with $\sigma(L) | \lambda(G)$.

The Scott topology $\sigma(\lambda(G))$ of the complete lattice $\lambda(G)$ is at least as fine as $\sigma(L) | \lambda(G)$, but it may be properly finer . The two topologies agree if for each $U \in \sigma(\lambda(G))$ the set $\uparrow_L U$ is in $\sigma(L)$. By C, p.181, Corollary 1.7 this is the case iff the kernel operator associated with $\lambda(G)$ is continuous. We therefore emphasize the following complement to Theorem 2.5:

2.7. COMPLEMENT TO THEOREM 2.5. The conditions of Theorem 2.5 imply each of the following:

(5) $\sigma(\lambda(G)) = \sigma(L) | \lambda(G)$,i.e. the topology of $\lambda(G)$ is the Scott topology (of $\lambda(G)$).

(6) For $a,b \in \lambda(G)$ we have $a \ll_{(G)} b$ iff $a \ll_L b$ iff there is a $c \in K(L)$ with $a \leq c \leq b$.

(Proof see C, p.181, Corollary 1.7). Condition (5) is a trivial consequence of (1). □

2.8.COMPLEMENT TO THEOREM 2.5. If , in Theorem 2.5 we have G = Irr L, then G is both inf - and sup - dense in $\lambda(G)$.
Proof. This follows from 1.7. □

3. Algebraic lattices and general topology.

Let X be an arbitrary T_o-space, then O(X) is a complete Heyting algebra with X as spectrum. We can form the complete Heyting algebra L=Filt O(X) of filters on O(X). Then L is in fact an algebraic Heyting algebra with an isomorphism $U \mapsto \uparrow U: O(X) \longrightarrow K(L)^{op}$, where $\uparrow U = \{V \in L: U \in V\}$ is the principal filter generated by U in L. Indeed L is arithmetic (cf. C, p.86, def. 4.6). The topology $\sigma(L)$ is generated by the basic sets $\uparrow(\uparrow U) \subseteq L$.

3.1 LEMMA. For each $x \in X$, the neighborhood filter $\mathcal{U}(x)$ is an element of Irr L.

Proof. This will follow from

3.2. LEMMA. Let $Y \subseteq X$ be a closed irreducible set. Then the filter $F(Y) = \{U \in O(X): U \cap Y \neq \emptyset\}$ is completely prime in L.

Proof. Among all filters on O(X) which do not contain $X \setminus Y$ there is a unique largest one, namely, F(Y). Thus F(Y) is maximal in $L \setminus \uparrow(\uparrow(X \setminus Y))$. Hence $F(Y) \in$ Irr L by C, p.92, Proposition 4.21. Better still: If p = max L k with an algebraic lattice L and a k K(L) with a prime k in $K(L)^{op}$, then p is completely prime. □

Indeed if $x \in X$, then $F(\{x\}^-) = \mathcal{U}(x)$, and thus 3.1 follows from 3.2. □

We denote the set of all neighborhood filters $\mathcal{U}(x)$ with G.

3.3. LEMMA. (Banaschewski). The function $x \mapsto \mathcal{U}(x): X \longrightarrow (G, \sigma(L)|G)$ is a homeomorphism.

Proof. See B.

In this fashion we may consider every T_o-space as a subspace of (Irr L, $\sigma(L)|$Irr L) for an arithmetic lattice L.

3.4. LEMMA. The subset G of L is K-generating (2.1).

Proof. Let $c \in K(L)$. Then there is a $U \in O(X)$ with $c = \uparrow U$. The Lemma follows since

$$\uparrow U = \bigcap \{\mathcal{U}(x): x \in U\}. \ \square$$

3.5. CONVENTION. Under the present circumstances we will denote the complete lattice and topological space $\lambda(G)$ (see 2.4 and 2.6) by λX. □

3.6. THEOREM. (Banaschewski). The map $x \mapsto \mathcal{U}(x): X \longrightarrow \lambda X$ is the (unique) essential hull of X.

Proof. See B. □

We say that X has an injective hull iff the space λX is injective in the sense of Scott and C iff $\lambda X = \lambda(G)$ is a continuous lattice with $\sigma(\lambda(G)) = \sigma(L)|\lambda(G)$. A translation of the conditions of Theorem 2.5 allows us to characterize the spaces X which have an injective hull.

We need the formalism of the specialisation order, which we record for the sake of completeness:

3.7. DEFINITION. We associate with a T_o-space X two transitive relations:

(i) The specialisation order $\leqslant$ given by $x \leqslant y$ iff $x \in \{y\}^-$ iff $\mathcal{U}(x) \subseteq \mathcal{U}(y)$.

(ii) The Scott order $\prec$ given by $x \prec y$ if $y \in \text{int} \uparrow x$, where $\uparrow x$ denotes the upper set of x w.r.t. the specialisation order.

Following Banaschewski we write for any subset A of X

$$\Gamma_o(A) = \bigcap\{\{a\}^- : a \in A\} = \bigcap\{\downarrow a : a \in A\} = \{x \in X : U \in \mathcal{U}(x) \text{ implies } A \subseteq U\}.$$

If V is an open subset of X, then $\Gamma_o(V) = \{x \in X: \mathcal{U}(x) \subseteq \uparrow V\} = \{x \in X: V \subseteq \uparrow x\} = \bigcap\{\downarrow w: w \in V\}$.

With the specialisation order $\leqslant$ we associate as usual a thrird order:

(iii) The way below relation $x \ll y$ (see C,p.38, Def. 1.1) □

The relation $x \prec y$ implies $x \ll y$, the converse fails even in complete lattices with the Scott topology (see C,p.111, Ex.1.25). The equivalence of the two relations means that the sets $\Uparrow x$ are all open in X.

3.8. THEOREM. Let X be a T_o-space. Then the following conditions are equivalent:

(1) X has an injective hull.

(2) For each $x \in X$, the neighborhood filter $\mathcal{U}(x)$ is the sup in the filter lattice Filt O(X) of all $\mathcal{U}(y)$ with $y \prec x$.

(3) For each $x \in X$ and each completely irreducible filter $\mathcal{U}$ not containing $\mathcal{U}(x)$ there is a point z with $\mathcal{U}(z) \not\subseteq \mathcal{U}$ such that for some open neighborhood V of x we have $z \in \Gamma_o(V)$ (i.e. $\mathcal{U}(z) \subseteq \uparrow V$)

(4) For each point $x \in X$ and each open neighborhood W of x there is an open neighborhood V of x and finitely many points $x_1,\ldots,x_n$ such that $x_1,\ldots,x_n \in \Gamma_o(V)$ and $W_1 \cap \ldots \cap W_n \subseteq W$ for suitable neighborhoods W_j of x_j (i.e., such that $W \in \mathcal{U}(x_1) \vee \ldots \vee \mathcal{U}(x_n) \subseteq \uparrow V$).

Proof. For two points x and y of X, the following three statements are equivalent:

(i) there is an open set V such that $\mathcal{U}(x) \subseteq \uparrow V \subseteq \mathcal{U}(y)$,

(ii) there is an open set V such that $x \in \Gamma_o(V)$ and $y \in V$,

(iii) $x \prec y$.

Clearly 2.5(1) is equivalent to (1) above. In the light of our preceding remarks, 2.5(2) is equivalent to (2) above. Hence (1) $\Leftrightarrow$ (2). Condition (3) above is a translation of 2.5(3), and condition (4) above is a reformulation of 2.5(4) in the present circumstances. □

3.9. COMPLEMENT TO THEOREM 3.8. The conditions of Theorem 3.8 imply each of the following:

(5) For each $x \in X$ we have $x = \sup \twoheaddownarrow x = \sup \Downarrow x$.

(6) The topology of λX is the Scott topology.

(7) For $\mathcal{U},\mathcal{V} \in \lambda X$ we have $\mathcal{U} \ll_{\lambda X} \mathcal{V}$ iff there is an open set $V \in \mathcal{V}$ such that $\mathcal{U} \subseteq \uparrow V$.

Proof. Since the specialisation order of X is induced by the order of L = Filt O(X), condition (5) is a consequence of (2), considering that $\twoheaddownarrow x \subseteq \Downarrow x$ by a remark following 3.7.
(6) and (7) follow by 2.7. □

3.10. <u>COMPLEMENT TO THEOREM</u> 3.8. Assume, in addition to the hypotheses of 3.8, that every completely irreducible element of Filt O(X) is a neighborhood filter of a point. Then the following conditions are equivalent:

(1) X has an injective hull.

(8) For each point $x \in X$ and each point y with $x \not\leq \{y\}^-$ there is a point $z \not\leq \{y\}^-$ and an open neighborhood V of x such that $z \in \Gamma_o(V)$.

(8') For each point $x \in X$ and each point y with $x \not\leq y$ there is a $z \not\leq y$ with $z \prec x$.

(9) For each $x \in X$ we have $x = \sup \twoheaddownarrow x$ in X.

Moreover, under these circumstances, (the image of) X is inf- and sup dense in λX.
Proof. Under the present conditions, (8) is equivalent to 3.8(3), and (8) is equivalent to (8'). By 3.9 we know that (1) implies (9) Remains to show that (9) implies (1). First we observe that the present hypotheses mean that G = Irr L for L = Filt O(X). Thus 2.8 applies and shows that X is inf- and sup-dense in λX (if we identify X with its image via 3.3 in λX). Lemma 0.3 on p.9 of A(5) (R.-E.Hoffmann) applies to show that under these circumstances, the embedding $X \to \lambda X$ preserves all existing sups. The inclusion $\lambda X \to L$ preserves sups anyhow. Thus the embedding $X \to L$ preserves all existing sups. But then condition (9) above implies 3.8(2). (A direct proof of (8') $\Leftrightarrow$ (9) is also possible.) □

3.11. <u>REMARK</u>. A sufficient condition that for a T_o-space X every completely irreducible filter on O(X) is a neighborhood filter is that X be sober and Filt O(X) be join-continuous.

Proof. If L is an algebraic lattice such that L is join-continuous and distributive, then every $p \in$ Irr L is uniquely determined by a $c(p) \in K(L)$ such that $c(p) = \min L \setminus \downarrow p$, where c(p) is a prime of $K(L)^{op}$; conversely, every prime c of $K(L)^{op}$ yields a $p(c) \in$ Irr L given by $p(c) = \max L \setminus \uparrow c$. (See C, p.92, Proposition 4.21 and LNM 369 (1974) p.60, Corollary 1.15.). Thus if L = Filt O(X), the completely irreducible elements of L are those filters which are determined by a prime element U of O(X) as maximal filters not containing U; since X is sober, $U = X \setminus \{x\}^-$ for some x and the maximal filter not containing U is $\mathcal{U}(x)$. □

(What we are saying here is that in a join continuous algebraic lattice, every completely irreducible element is completely prime.)

Theorem 3.6 replaces the incorrect Corollary 2 to Proposition 3 in Banaschewski's paper B. The mistake in his proof appears on p.239 ,line 7 from the bottom, where the unions must be replaced by suprema.

3.12. PROPOSITION. (Banaschewski). A T_1-space has an injective hull iff it is discrete.

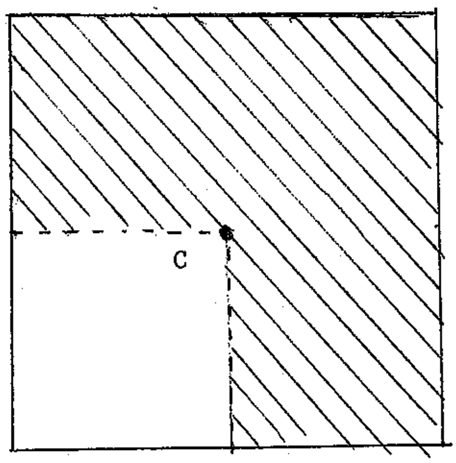
Proof. This follows from 3.8(4)

3.13. EXAMPLE. The boundary of the square in its Scott topology has the square as injective hull. (See D). The open subset of all (x,y) with $y=1$ and $x>0$, or $x=1$ and $y>0$ does not have an injective hull by 3.8. □

This example shows that Corollary 4 in B on p 240 is false.

The following example is instructive:

3.14. EXAMPLE. Consider the following subset of the square:



The space arising from this set by endowing it with the topology induced from the Scott topology of the square we call X. The space $X_o = X \setminus \{C\}$ carries its own Scott topology and was discussed in D. The embedding into the square $X_1 = [0,1]^2$ (with the Scott topology) is the essential hull of X_o as was shown in D. By Banaschewski's Lemma 2 on p. 235 of B we then know that the embedding of X into X_1 is the injective hull of X_1, since $X_o \to X \to X_1$ is a sequence of embeddings. Thus X has an injective hull. (It is instructive to verify explicitly the equivalent conditions of 3.8.) We also know from 3.9 that the Scott topology of the square is the topology induced from the Scott-topology of Filt O(X). But we observe:

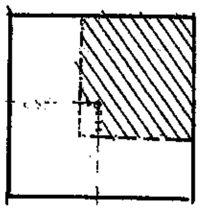
(i) X is not locally quasicompact at C.

(ii) X is sober.

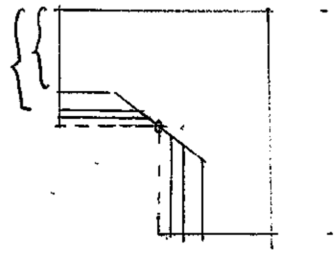
(iii) The Scott topology of X (w.r.t. the specialisation order) is finer than the given topology. It is locally quasicompact sober and quasicompact.

(iv) $X'=(X, \sigma(X, \leq))$ does not have an injective hull.

Proof.i)Each neighborhood of C must contain a neighborhood of the form

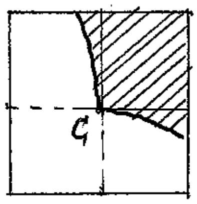


This allows us to cover any neighborhood with a countable ascending sequence of open sets,none of which covers the neighborhood.



ii) A closed irreducible set is directed w.r.t. the specialisation order. By inspection we note that all those sets have a maximum.

iii) The point C has a basis of Scott-open neighborhood of the form:



Now C has a basis of such neighborhoods which are closed in the topology of the square; these are still quasicompact w.r.t. $\sigma(X, \leq)$. Thus $X'=(X, \sigma(X, \leq))$ is locally quasicompact sober and quasicompact. The specialisation order of this space is the natural order of the square and thus agrees with that of X.

iv) We show that 3.8(2) is violated. The only problematic point is C. In X' we have $(0,y) \prec C$ iff $y \prec 1/2$; and $(x,0) \prec C$ iff $x < 1/2$. We DO NOT HAVE $C \prec C$. But then the sup of all $\mathcal{U}(t)$ with $t \prec C$ is the neighborhood filter of C w.r.t. $\sigma(X_1)|X'$ which is bigger than $\mathcal{U}(C)$.

We notice the following additional information about this example:

(v) The point C does not satisfy $C \ll C$ even though C is isolated from below.

This is a subtle point. Suppose we define $x <' y$ by the following relation:

Whenever $y = \sup D$ for a directed set D, then $x \leq d$ for some $d \in D$.

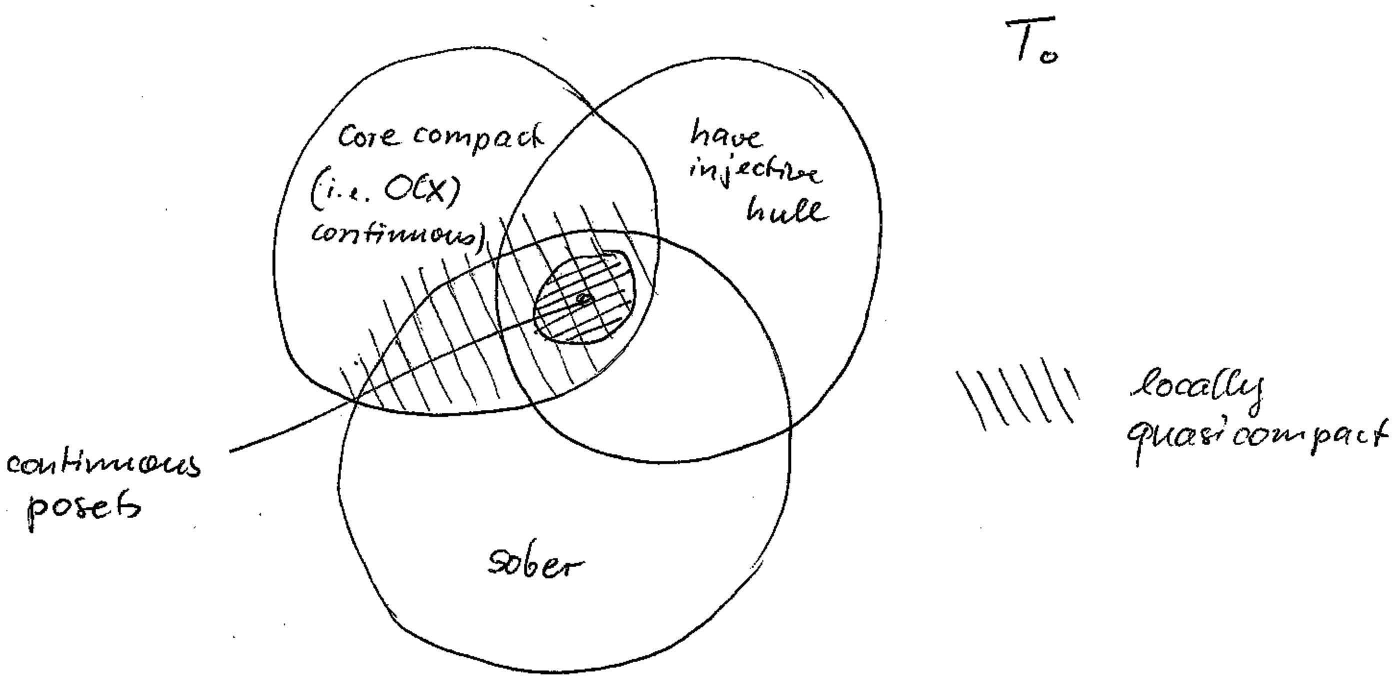
Then $C <' C$ and X' satisfies the following two conditions:

(a) Every set $\{y : y <' x\}$ is directed for all x.

(b) $x = \sup \{y : y <' x\}$ for all x.

This means that X' is as close to a continuous poset (it is, of course, up-complete!) as can be without being one. □

The following diagram illustrates the relationship between various classes of spaces which arise in the present context. The point is that no general relation exists.



However, it is useful to juxtapose core compactness and the property of having an injective hull. This is most conveniently achieved by focussing on condition 3.8(4) which we repeat for the sake of self sufficiency and comparison:

(IH) $(\forall x \in X)(\forall W \in \mathcal{U}(x))(\exists V \in \mathcal{U}(x))(\exists x_1,\ldots,x_n)\ W \in \bigvee_{j=1}^{n} \mathcal{U}(x_j) \subseteq \uparrow V$.

On the other hand, a space is core compact (i.e. O(X) is continuous) iff

(CC) $(\forall x \in X)(\forall W \in \mathcal{U}(x))(\exists V \in \mathcal{U}(x))(\exists \mathcal{H})\ W \in \mathcal{H} \subseteq \uparrow V$ and $\mathcal{H}$ is Scott open in O(X).

(cf. C. p.131)

It follows that a T_o-space with an injective hull is core compact whenever a finite sup of neighborhood filters is a Scott open set on $O(X)$. We recall that the neighborhood filter of any quasicompact set is Scott open. One conclusion which one may draw from this observation is the following:

3.14. PROPOSITION. Let X be a T_o-space satisfying the following conditions:

(i) X has an injective hull.
(ii) The specialisation order turns X into a topological sup-semilattice.

Then X is core compact.

Proof. We claim that (ii) implies that $\mathcal{U}(x_1) \vee \ldots \vee \mathcal{U}(x_n) = \mathcal{U}(x_1 \vee \ldots \vee x_n)$:
The containment $\subseteq$ is clear. In order to show the reverse containment let U be a neighborhood of $x_1 \vee \ldots \vee x_n$. Since $\vee$ is continuous by (ii), there are open neighborhoods U_j of x_j with $U_1 \cap \ldots \cap U_n = U_1 \vee \ldots \vee U_n \subseteq U$. □

Another class of spaces which arose in 3.10 and 3.11 is not yet sufficiently clarified. Let us assume for simplicity that X is a SOBER space. Then the set G of neighborhood filters in L = Filt O(X) is precisely the set of completely prime elements of L. When is G = Irr L ? This is the case for an algebraic lattice L, iff the complete lattice characters $L \longrightarrow 2$ separate the points, iff L is completely distributive, iff L is join continuous. (I think.) The question then becomes the following: If K is a complete lattice, which conditions on K will make Id K completely distributive?

The answer is simple: In Hofmann, Mislove and Stralka, Lecture Notes in Mathematics, <u>396</u> (1974), pp. 69 ff., notably Theorems 1.33 and 1.37 we saw that an algebraic lattice L is completely distributive iff $K(L)^{op}$ is algebraically generated by its prime elements (i.e. every element in $K(L)^{op}$ is a finite inf of primes). If L = Filt O(X) (of X), then L is completely distributive iff every open set is a finite intersection of prime open sets, i.e., every closed set is the finite union of closed irreducible sets. Such spaces have been called Noetherian. Thus in the case of Noetherian spaces, 3.1o and 3.11 apply.