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TOPIC A strict extension of previous results on essential extensions

REFERENCE The uncollected works of R.-E. Hoffmann, in particular,

[$H_I$] Essentially complete $T_0$-spaces, I

[$H_{II}$] Essentially complete $T_0$-spaces, II

[B] B. Banaschewski, Essential extensions of $T_0$-spaces

[HM] Hofmann & Mislove, SCS 6-8-82 and SCS 5-28-82

Let P be a poset and assume that P is embedded in a complete lattice L. Then one can obtain the MacNeille completion of P by first forming $L_1$, the set of all infs of subsets of P (equivalently the lattice order-generated by P), and then forming $L_2$, the set of ll sups of subsets of P in the complete lattice $L_1$. The inclusion of P into $L_2$ is then well-known to be the MacNeille completion.

While perusing the recent SCS memos of Hofmann & Mislove, I began to wonder if perhaps some similar phenomenon didn't hold in regard to the essential extension of a topological space. I was particularly struck by the theorem of R.-E. Hoffmann quoted in the earlier memo concerning when a continuous lattice is the essential hull of an embedded continuous poset and the computations of the essential hull contained in the memo. I decided that the viewpoint there was too microscopic (as opposed to Karl's usual telescopic stance) and that there must be some better, more general approach.

Let X be a $T_0$-space which is topologically embedded in - continuous lattice L equipped with its Scott topology. (There are a variety of ways of doing this. Recall for example that any $T_0$-space can be embedded in a product of 2's by using the characteristic maps of open sets. In [B] a space is embedded into the filter lattice of the lattice of open sets, an algebraic lattice as Hofmann & Mislove point out in their second memo.) Let $L_1$ be the continuous sublattice generated by X in L. Let $L_2$

be the set of all sups of subsets of X in the lattice $L_1$. Then the inclusion of X into $L_2$ endowed with the relative Scott topology gives the essential hull of X. This is the essential result of this memo and what we now set about to prove. Note that this result easily gives that the square is the essential hull of the examples considered by HM in their first memo. Note also the similarity of this situation and the case of the MacNeille completion (where order generation in the first step is replaced by generation).

We turn now to the proof of the above assertion. As has become apparent in some of Hoffmann's work, the situation concerning the 1 requires a little special care (particularly with respect to the notion of generation).

Unfortunately I have not seen R.-E. Hoffmann's latest preprints, so I am not sure to what extent the following results may overlap his work.

1. DEFINITION. Let $\emptyset \neq A \subseteq L$, a continuous lattice. The $\wedge$-semilattice generated by $A$ is denoted $\langle A \rangle$. The CL-subobject generated by $A$ is $\overline{\langle A \rangle}^{\lambda} \cup \{\sup A\}$ (where $\overline{\phantom{x}}^{\lambda}$ denotes the closure in the CL- or $\lambda$-topology). (Note that it may or may not be the case that $\sup A \in \overline{\langle A \rangle}^{\lambda}$.) We say $A$ generates $L$ if $L$ is the CL-subobject generated by $A$. ▯

2 REMARK. The CL-subobject generated by $A$ consists of closing up $\langle A \rangle$ under directed infs, then directed sups, then add $\sup A$ (if necessary).

3 LEMMA. Let $X$ generate $L$, where $L$ is a continuous lattice. If $U$ is Scott-open, then $\inf(U \cap X) = \inf(U)$.

Proof. If $U = \emptyset$ ~~or $U = \{1\}$~~, then $\inf(U \cap X) = 1 = \inf(U)$. Otherwise $\inf U = \inf(U \setminus \{1\}) = \inf(U \setminus \{1\} \cap \langle X \rangle) =$

(since $U\setminus\{1\}$ is $\lambda$-open and $\langle X\rangle$ is $\lambda$-dense in $L\setminus\{1\}$)

$= \inf(U \cap \langle X\rangle) = \inf(U \cap X)$ (since $U = \uparrow U$). $\square$

Let $i: X \to Y$ be an embedding of $T_0$-spaces. Then $i_*: O(Y) \to O(X)$ defined by $V \mapsto i^{-1}(V)$ is a lower adjoint with upper adjoint $i^*: O(X) \to O(Y)$ defined by $i^*(U) = U^* = \bigcup\{V \in O(Y) : U = i^{-1}(V)\}$, i.e. $U^*$ is the largest open set in $Y$ such that $i^{-1}(U^*) = U$. Recall from [B] that the embedding $i$ is <u>strict</u> if $\{U^* : U \in O(X)\}$ forms a basis for the topology of $Y$.

4 PROPOSITION. Let $L$ be a continuous lattice generated by $X$. Assume that $L$ is equipped with the Scott topology and $X$ with the subspace topology. Then the inclusion $i: X \to L$ is strict.

Proof. Let $x \in V$ where $V$ is Scott-open. Then $\exists\, z \ll x$ such that $z \in V$. Let $\mathcal{U} = (\hat{\uparrow} z \cap X)^*$. Then $\mathcal{U} \cap X = \hat{\uparrow} z \cap X$. Thus $\inf \mathcal{U} = \inf(\mathcal{U} \cap X)$ (by Lemma 3) $= \inf(\hat{\uparrow} z \cap X) \geq z$. Thus we have $x \in \hat{\uparrow} z \subseteq \mathcal{U}$ (by definition of $i^*$) $\subseteq \uparrow z \subseteq V$. Since $x$ and $V$ were arbitrary, $i$ is strict. □

We recall further from [B] that an embedding $i: X \to Y$ is (i) <u>essential</u> if given any continuous $f: Y \to Z$ in $TOP_0$ such that $f \circ i$ is an embedding, then $f$ is an embedding; (ii) <u>superstrict</u> if given any collection $\mathcal{B} \subseteq \mathcal{O}(Y)$ closed under finite intersections such that $\mathcal{B}|X$ is a basis for $X$, then $\mathcal{B}$ is itself a basis for $Y$.

By Proposition 1 of [B] $i$ is superstrict iff it is essential. We shall need only the easily verifiable implication that superstrict implies essential.

5 LEMMA. Let $L$ be $T_0$-space satisfying:

(i) The associated partial order $(L, \le)$ is a complete lattice.

(ii) The topology on $L$ is contained in the Scott topology of $(|L|, \le)$.

(iii) $\vee : L \times L \to L$ sending $(x, y) \to x \vee y$ is (jointly) continuous.

Let $X$ be a subspace of $L$ such that $j : X \hookrightarrow L$ is strict. Let $Y = \{\sup A : A \subseteq X\}$ endowed with the relative topology from $L$. Then $i : X \hookrightarrow Y$ is essential.

6 NOTE. By Proposition 1.1 of $[H_1]$ the $T_0$-spaces satisfying (i), (ii), and (iii) are precisely the essentially complete $T_0$-spaces. A continuous lattice endowed with the Scott topology satisfies these conditions. $\square$

Proof (of Lemma 5). Let $\mathcal{B}$ be a collection of open subsets of $Y$ closed under finite intersection such that $\mathcal{B}|X$ is a basis for $X$. If we show this implies that $\mathcal{B}$ is a basis for $Y$, then $i: X \hookrightarrow Y$ will be superstrict and hence essential.

Let $y \in Y$ and let $V$ be an open set (in $L$) containing $y$. If $y = 0$, then $V = L$, so $V \cap Y = Y$, which is the intersection of the empty collection and hence in $\mathcal{B}$.

If $y \neq 0$, then $y = \sup A$ where $\emptyset \neq A \subseteq X$.

Since by (ii) $V$ is Scott open, $\exists$

$F = \{x_1, \ldots, x_n\} \subseteq A$ such that $\sup F \in V$.

By continuity of $V$, there exist open sets

$V_i$ such that $x_i \in V_i$ for $1 \leq i \leq n$ and

$\bigcap_{i=1}^{n} V_i \subseteq V$. Since $j : X \hookrightarrow L$ is strict,

there exist sets $U_i$, $i=1, \ldots, n$, open in $X$

such that $x_i \in j^*(U_i) \subseteq V_i$. Then

$x_i \in j^*(U_i) \cap X = U_i$. Since $\mathcal{B} | X$ is a

basis, $\exists$ $B_i \in \mathcal{B}$, $1 \leq i \leq n$, such that $x_i \in B_i \cap X \subseteq U_i$.

Then $B = \bigcap_{i=1}^{n} B_i \in \mathcal{B}$ by hypothesis, Furthermore

$x_i \leq y$ for all $i \Rightarrow y \in B_i$, $1 \leq i \leq n \Rightarrow y \in B$.

Finally $B_i \subseteq i^*(B_i \cap X) \subseteq j^*(B_i \cap X) \subseteq j^*(U_i) \subseteq V_i$

for $1 \leq i \leq n$. Thus $y \in B = \bigcap_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n} V_i \subseteq V$. $\square$

7 THEOREM. Let $X$ be a subspace of a continuous lattice $L$ endowed with the Scott topology. Let $L_1$ be the CL-subobject generated by $X$ and let $\lambda X = \{\sup A : A \subseteq X\}$ where sups are taken in the complete lattice $L_1$. Then the inclusion of $X$ into $\lambda X$ endowed with the relative topology gives the essential hull of $X$.

Proof. By the Compendium the relative topology of $L_1$ is the Scott topology. By Proposition 4 the inclusion from $X$ into $L_1$ is strict. By Lemma 5 and Note 6 the inclusion from $X$ into $\lambda X$ is essential. Now $\lambda X$ still satisfies (i), (ii), and (iii) of Lemma 5, hence is essentially complete by Proposition 1.1 of [$H_I$]. Thus $\lambda X$ is the essential hull of $X$. ▯

8 COROLLARY. If a subset $X$ of a continuous lattice $L$ generates and order cogenerates, then $\Sigma L$ is the essential hull of $X$ equipped with relative Scott topology. In particular, $X$ has an injective hull.