

Corrected Copy J H Cornuth

STILL MORE NOTES ON CHAINS IN CL-OBJECTS (from JHC) 6/15/76

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REFERENCES: Notes on chains in CL-Objects (from KHH) 4/19/76

More notes on chains in CL-Objects (from JHC) 5/23/76

Well, it seems as though someone will need to supply some additional insight into this problem . . . that is, the canonical maximal strict chain problem. I will, however, outline our approach in hopes that it might be useful to someone. First, our approach is not to throw anything out systematically, but rather to start with almost nothing and throw things in systematically, thus attempting to construct such maximal strict chains. Admittedly, "canonical" is in the eyes of the beholder, but until someone comes up with something better, here goes:

Throughout,  $T$  is a fixed compact chain in a CL-object  $S$ .

Definition 1.  $x \in T$  is a left [right] good point of  $T$  if  $\forall y \in T, x < y$  implies  $x \ll y$  [ $\forall y \in T, y < x$  implies  $y \ll x$ ].

$x$  is a good point of  $T$  if it is both a left good point of  $T$  and a right good point of  $T$ . We denote by  $[L, R]$   $G$  the set of [left, right] good points of  $T$ .

Proposition 1. (1)  $G = L \cap R$ .

(2)  $\inf T \in R$ .

(3)  $\sup T \in L$ .

(4)  $L$  is "lower closed", that is if  $x = \inf [(\uparrow x) \cap L]$  then  $x \in L$ .

(5)  $R$  is "upper closed".

Proof. (1), (2), and (3) are transparent and (5) is dual to (4).

For (4), if  $x = \inf [(\uparrow x) \cap L]$  and  $x < y$  in  $T$ , then there exists  $a \in L$  such that  $x \leq a < y$ . But then  $x \leq a \ll y$  (since  $a \in L$ ) and it follows that  $x \ll y$  <sup>\*</sup>. Hence  $x \in L$ .

Observation 1. If  $C$  is any strict chain which is maximal with respect to being contained in  $T$ , then  $G \subseteq C$ .

Observation 2.  $L \cup R$  is a strict chain containing  $G$ .

Proposition 2.  $L \cup R$  is closed.

Proof. Suppose  $x = \sup A$  where  $A \subseteq L \cup R$  and  $x \notin A$ .

Fix  $y \in T$  such <sup>that</sup>  $y < x$ . Then there exists  $z \in A$  such that  $y < z < x$ . If  $z \in L$ , then  $y \leq z \ll x$  and so  $y \ll x$  (see the table on page 4 of reference 2) <sup>\*</sup>. If  $z \in R$ , then  $y \ll z \leq x$  and so  $y \ll x$  (again, see the table on page 4 of reference 2) <sup>\*</sup>. Hence,  $x \in L \subseteq L \cup R$ . Dually, if  $x = \inf A$  where  $A \subseteq L \cup R$  and  $x \notin A$ , we obtain  $x \in R \subseteq L \cup R$ .

Unfortunately,  $L \cup R$  need not be a strict chain so we have tossed in too much. However, our next proposition tosses out just enough of  $L \cup R$  to recover the strict chain property.

Proposition 3. The following subset of  $L \cup R$  is a strict chain:

$$D = (L \cup R) \setminus \{y : \exists x \in (L \cup R) \ni [x, y]_{L \cup R} = \{x, y\}, x < y, x \not\ll y\}.$$

Proof. For this result we will need a few preliminary results.

Lemma 1. If  $[x, y]_{L \cup R}$  is an  $L \cup R$  gap, then TAE:

- (a)  $y \in D$ .
- (b)  $x \ll y$ .

If these equivalent conditions do not hold then  $x \in D$ .

Proof of Lemma 1: That (b) implies (a) is immediate

from the definition of  $D$ ,  $y$  simply doesn't get tossed out. If (b) does not hold, then  $y$  does get tossed out and so (a) does not hold. This shows that (a) and (b) are equivalent. Finally, if the conditions do not hold then  $x \not\ll y$  and so  $x \notin L$ . This implies that  $x \in R$  and our next Lemma yields that  $R \subseteq D$ .

Lemma 2.  $R \subseteq D$ .

Proof of Lemma 2: If  $y \in R$  and  $x < y$ , then  $x \ll y$ .

Hence  $y$  has no chance of being removed in building  $D$ .

Corollary 1. If  $[x, y]_{L \cup R}$  is an  $L \cup R$  gap, then  $x \in D$  or  $y \in D$ .

Proof of Proposition 3 (I just found a proof which does not use any of the preliminary results. However they will come in handy anyway.)

Fix  $x < y$  in  $D$ . If there exists  $z \in L \cup R$  such that  $x < z < y$ , then it follows that  $x \ll y$  as in the proofs of Proposition 1 (4) and Proposition 2. If there does not exist  $z \in L \cup R$  such that  $x < z < y$ , then  $[x, y]_{L \cup R}$  is an  $L \cup R$  gap and so  $x \ll y$  (for otherwise  $y$  would not be in  $D$ ).

Our next proposition shows that  $D$  is a maximal strict chain relative to being contained in  $L \cup R$ .

Proposition 4. If  $C$  is a strict chain in  $S$  and  $D \subseteq C$ ,  
then  $D = C \cap (L \cup R)$ .

Proof. Fix  $y \in C \cap (L \cup R)$ . If  $y \notin D$ , then there exists  $x \in L \cup R$  such that  $[x, y]_{L \cup R}$  is an  $L \cup R$  gap,  $x < y$ ,  $x \not\ll y$ . By Lemma 1,  $x \in D \subseteq C$ , so  $x \ll y$ . Hence, the assumption that  $y \notin D$  is false.

Observation 3. If  $t \in T \setminus D$  and  $D \cup \{t\}$  is a strict chain, then  $t \notin L \cup R$ . Hence, if this be the case we have that  $x = \sup \{w \in L \cup R : w < t\} < t$  (since  $x \in L \cup R$  by Proposition 2) and  $t < y = \inf \{w \in L \cup R : t < w\}$  (again since  $y \in L \cup R$ ).

The unshot of Observation 3 is that it now suffices to consider intervals in  $T$  of the form  $(x, y)_T$  where  $[x, y]_{L \cup R}$  is an  $L \cup R$  gap and somehow find a canonical way to toss in enough points to obtain a strict chain which is maximal with respect to being contained in  $[x, y]_T$ . Now, since none of the points in  $(x, y)_T$  are "good", it seems reasonable to delineate various types of bad points.

Definition 2.  $s \in T$  is a left [right] bad point of  $T$  if there exists  $t \in T$  such that  $s < t$  and  $s \not\ll t$  [ $t < s$  and  $t \not\ll s$ ].  $s$  is a bad point of  $T$  if it is both a left bad point of  $T$  and a right bad point of  $T$ .  $s$  is a very bad (perhaps "way bad"?) ((joking of course)) point of  $T$  if there exist  $r, t \in T$  such that  $r < s < t$ ,  $r \not\ll s \not\ll t$ , and  $r \not\ll t$ .

At this point we were faced with the chore of attempting to define "corner points of  $T$ ". In  $I^2$  this is quite easy. They are simply the bad points of  $T$  which are not very bad points of  $T$ . That is: if  $T \subseteq I^2$ , then we can define  $s$  to be a "corner point<sub>1</sub>" if  $s$  is a bad point of  $T$  and  $\forall r, t \in T$  such that  $r < s < t$  and  $r \not\ll s \not\ll t$  we have  $r \ll t$ . It is easy to show that  $s$  is a

"corner point<sub>1</sub>" iff  $s$  is a bad point of  $T$  and  $\forall r, t \in T$  such that  $r < s < t$  we have  $r \ll t$ .

Observation 4. Recall that we are attempting to toss points of  $(x, y)_T$  in with  $D$  to obtain a strict chain which is maximal relative to being contained in  $[x, y]_T$ . Certainly if  $x \not\ll y$ , nothing can be added. Hence, we may as well assume that  $x \ll y$  and, for that matter, we will also assume that  $(x, y)_T \neq \square$ .

Proposition 5. If  $T \subseteq I^2$ , then  $(x, y)_T$  contains at least one "corner point<sub>1</sub>".

I can prove this proposition but it seems to be of little consequence other than the fact that it does allow one to construct appropriate strict chain in a rather canonical fashion for chains  $T \subseteq I^2$ .

To do this one considers the three possibilities: (these are exhaustive)

- (i) There are finitely many "corner point<sub>1</sub>"s in  $(x, y)_T$ ;
- (ii) The set of "corner point<sub>1</sub>"s is an increasing sequence converging to  $y$ ;
- (iii) The set of "corner point<sub>1</sub>"s is isomorphic to  $\mathbb{Z}$  with min multiplication, with  $\mathbb{Z}^+$  converging to  $y$  and with  $\mathbb{Z}^-$  converging to  $x$ .

Unfortunately, "corner point<sub>1</sub>"s need not exist if  $T \not\subseteq I^2$ . In fact one only needs to go to  $I^3$ . For convenience, we will let  $I = [0, 4]$ .

Let  $0 = (0,0,0)$ ,  $x = (1,0,0)$ ,  $a = (1,1,0)$ ,  $b = (1,1,1)$ ,  
 $c = (2,1,1)$ ,  $d = (2,2,1)$ ,  $e = (2,2,2)$ ,  $y = (3,2,2)$ , and  $1 = (4,4,4)$ .  
 Let  $T = \{0, x, a, b, c, d, e, y, 1\}$ . Then  $L \cup R = \{0, x, y, 1\} = D$ , and  
 $(0, x)_T = \square = (y, 1)_T$ . So,  $(x, y)_T$  is the only interval  
 deserving attention. Note that  $C = \{0, x, c, y, 1\}$  is the unique  
 strict chain which is maximal relative to being contained in  $T$ .  
 However,  $c$  is not a "corner point<sub>1</sub>".

So, our next attempt was to define another type of "corner point"  
 as follows:  $c \in T$  is a "corner point<sub>2</sub>" if

- (i)  $c$  is a bad point of  $T$ ;
- (ii) if  $p = \inf \{t \in T : t < c \text{ and } t \neq c\}$ , then  
 $c = \sup \{t \in T : p < t \text{ and } p \neq t\}$ ; and
- (iii) if  $q = \sup \{t \in T : c < t \text{ and } c \neq t\}$ , then  
 $c = \inf \{t \in T : t < q \text{ and } t \neq q\}$ .

However,  $b$  and  $d$  are also "corner point<sub>2</sub>"s ~~and~~ and it is difficult  
 to decide why  $c$  must be picked. In fact, by changing  $T$  slightly  
 we find that "corner point<sub>2</sub>"s exist, a point must be added to  $D$   
 to obtain a strict chain which is maximal relative to being contained  
 in  $T$ , and yet neither of the possible points that can be added  
 is a "corner point<sub>2</sub>". To see this, let  $c_1 = (1.5, 1, 1)$  and let  
 $c_2 = (2, 1.5, 1)$ . Let  $T = \{0, x, a, b, c_1, c_2, d, e, y, 1\}$ . Again,  
 $L \cup R = \{0, x, y, 1\} = D$ ,  $b$  and  $d$  are the only "corner point<sub>2</sub>"s,  
 and the only strict chains which are maximal relative to being  
 contained in  $T$  are  $C_1 = \{0, x, c_1, y, 1\}$  and  $C_2 = \{0, x, c_2, y, 1\}$ .

This concluded our attempts to define "corner points".

Our final (last gasp) approach was to try to "construct the chains inductively" for  $T \subseteq \mathbb{I}^n$  by looking successively at maximal strict chains in the intersection of  $T$  with hyperspaces<sup>planes</sup>. This also failed.

Actually we tried several other notions of "corner points" but the two mentioned herein seemed to come closest to yielding something.

ANY IDEAS? (other than the one we have already had - - namely, scrap the whole approach!)

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Dear Karl,

I realized last night that there is a mistake (at least one mistake I should say) in the "Still more notes on chains in CL-objects" dated 6/15/76. On page 5, there are four, rather than three, possibilities for "corner point<sub>1</sub>" configurations. I was remembering an earlier approach where  $y$  was in  $R$  which did not allow the fourth possibility. Hence, the additional possibility is:

- (iv) The set of "corner point<sub>1</sub>"s is a decreasing sequence converging to  $x$ .

Sorry about the oversight . . . hopefully there are no others.

Sincerely,

