SEMINAR ON CONTINUITY IN SEMILATTICES

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TOPIC: Freedom for completely distributive lattices (over continuous posets) ?

REFERENCES:

[Kah] Karl H. Hofmann: The category CD of completely distributive lattices and their free objects. SCS - Memo 11-24-81

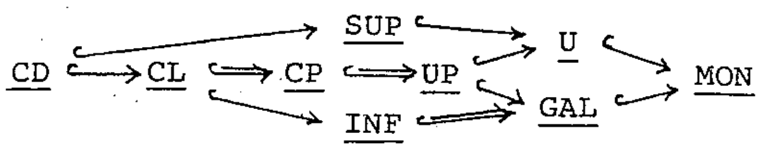
[Hom] Karl. H. Hofmann and M. Mislove: Free objects in the category of completely distributive lattices. Preprint Nr.676, TH Darmstadt 1982

Geben Sie Gedankenfreiheit
(F. Schiller, Don Carlos)

In [Kah] and [Hom] , K.H. Hofmann studied the following categories:

notation	objects	morphisms
MON	arbitrary posets	monotone maps
U	up-complete posets	Scott continuous maps
GAL	arbitrary posets	Galois maps (i.e. upper adjoints)
UP	up-complete posets	Scott continuous Galois maps
CP	continuous posets	"
CL	continuous lattices	"
SUP	complete lattices	sup-preserving maps
INF	complete lattices	inf-preserving maps
CD	completely distributive lattices	complete homomorphisms (i.e. sup- and inf-preserving maps)

These categories are related by the following hierarchy:



(⟹ means full subcategory)

For each up-complete poset P let T(P) denote the complete lattice of all Scott closed subsets of P . We can make T functorial by lifting any <u>U</u>- morphism $f : P \longrightarrow Q$ to a <u>SUP</u> - morphism

$$T(f) : T(P) \longrightarrow T(Q) , \; A \longmapsto \overline{f(A)}$$

where $\overline{}$ denotes the closure with respect to the Scott topology. It is easy to see that the maps

$$\eta_P : P \longrightarrow T(P) , \; x \longmapsto {\downarrow}x$$

are <u>U</u> - morphisms, and the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\;f\;} & Q \\ {\scriptstyle \eta_P}\downarrow & & \downarrow{\scriptstyle \eta_Q} \\ T(P) & \xrightarrow[T(f)]{} & T(Q) \end{array}$$

Let us recall some further facts from [Kah] .

LEMMA. *η_P has a lower adjoint (i.e. η_P is a <u>UP</u> - morphism) iff P is a complete lattice.*

THEOREM 1. *For each <u>UP</u> - morphism g from an up-complete poset P into a completely distributive lattice M , there exists a unique complete homomorphism $g^* : T(P) \longrightarrow M$ such that $g = g^*\eta_P$. Hence the restriction of T to <u>CL</u> is left adjoint to the forgetful functor from <u>CD</u> to <u>CL</u> ; the unit of the adjunction is given by η_P .*

Indeed, it is easy to see that g is Scott continuous iff g^* has an upper adjoint, and if $d : M \longrightarrow P$ is the lower adjoint of g then the map

$$d_* : M \longrightarrow T(P) , \; m \longmapsto \overline{d(\Downarrow m)}$$

is the lower adjoint of g^* , where

$$\Downarrow m = \bigcap \{ A = {\downarrow}A \subseteq M : m \leq \bigvee A \}$$

is Raney's "long way below set" (cf. [Kah, 2.6]).
For these conclusions, it is not necessary to assume that P be a complete lattice. But helas, in view of the Lemma, the above universal property does <u>not</u> provide a left adjoint for the forgetful functor from <u>CD</u> to <u>CP</u> . Kah conjectured that one might "tinker with the morphisms and improve the situation, but not very much can be done" (loc. cit.).
What can be done will be sketched on the following pages.

First of all, we must disappoint the reader who expects a satisfactory solution of the stated adjunction problem. insurmountable barriers are raised by the following

DILEMMA. *There is no category* $\underline{C}$ *whose objects are the continuous posets and whose morphisms are certain monotone maps, such that*
(1) $\underline{CD}$ *is a subcategory of* $\underline{C}$.
(2) The forgetful functor from $\underline{CD}$ *to* $\underline{C}$ *has a left adjoint with front adjunction* $\eta_P : P \longrightarrow T(P)$, $x \longmapsto {\downarrow}x$.
(In particular, for each continuous poset P, η_P *is a* $\underline{C}$ *- morphism).*

In spite of this Dilemma, it is possible to weaken the morphism concept in such a manner that each η_P becomes a "pseudomorphism" and for each "pseudomorphism" g from an up-complete poset P into a completely distributive lattice M, the map

$$g^* : T(P) \longrightarrow M \ , \ A \longmapsto \bigvee g(A)$$

becomes a complete homomorphism.

In order to define the required kind of maps, we single out a few typical properties of Galois maps. Let us call a map g between posets P and Q *quasiclosed* if $A \in T(P)$ implies ${\downarrow}g(A) \in T(Q)$ (cf. the notion of "quasiopen" in [Kah]). Further, we say g is a *pseudo - Galois map* provided that

$$g(Y_{\downarrow})^{\uparrow} = g(Y)_{\downarrow}{}^{\uparrow} \quad \text{for all } Y \subseteq P \ ,$$

where $Y_{\downarrow}$ and $Y^{\uparrow}$ denote the sets of all lower resp. upper bounds of Y . Finally, g is called a *weak Galois map* if

$$\overline{g(Y_{\downarrow})} = g(Y)_{\downarrow} \quad \text{for all } Y \subseteq P \ .$$

The position of these properties is analyzed in a

TRILEMMA. *(1) The Galois maps are precisely the quasiclosed weak Galois maps.*
(2) Every weak Galois map is a pseudo - Galois map.
(3) Every pseudo - Galois map preserves all existing infima.

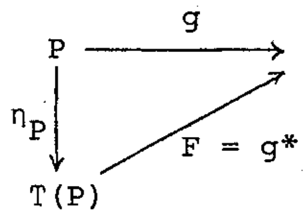
Now, by a *pseudomorphism* we mean a Scott continuous pseudo-Galois map, and by a *weak morphism* a Scott continuous weak Galois map. According to the Trilemma, every ($\underline{UP}$-)morphism is a weak morphism, and every weak morphism is a pseudomorphism. Suitable counterexamples show that none of these implications can be inverted. However, for maps between complete lattices all three notions coincide. Our main theorem states that pseudomorphisms have the same universal property as (UP-)morphisms, and they have the advantage that the natural embeddings η_P are always pseudomorphisms (while η_P fails to be a morphism unless P is complete).

THEOREM 2. *The following conditions are equivalent for a map* g *from an up-complete poset* P *into a completely distributive lattice* M :

(a) g *is a pseudomorphism.*

(b) *The map* $g^* : T(P) \longrightarrow M$, $A \longmapsto \bigvee g(A)$ *is a complete homomorphism.*

(c) *There exists a (unique) complete homomorphism* $F : T(P) \longrightarrow M$ *such that* $g = F\eta_P$.



Hence there is a one-to-one correspondence between the set of all pseudomorphisms $g : P \longrightarrow M$ *and the set of all complete homomorphisms* $F : T(P) \longrightarrow M$.

Notice that $\eta_P{}^*$ is the identity on T(P), so η_P is certainly a pseudomorphism.

Recall that for a *continuous poset* P , the system T(P) is a completely distributive lattice. Hence we derive from Theorem 2 the following

FACT. *A map* g *from a continuous poset* P *into a completely distributive lattice* M *is a pseudomorphism iff* g^* *is a* CD*-morphism.*

At first glance, this seems to be precisely what we need for an adjoint situation between the functor T and a forgetful functor in the converse direction. The only reason why this adjunction does not work is a little (but essential)

DEFECT. *The composition of two pseudomorphisms is in general not a pseudomorphism.*

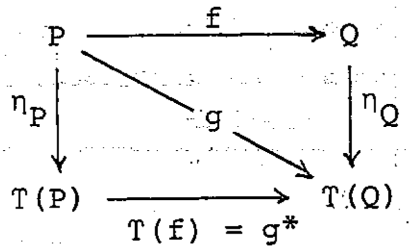
A simple counterexample is obtained by taking for P a two-element antichain and considering the composition g of the pseudomorphisms η_P and $\eta_{T(P)}$. Here we have $g(P_\downarrow)^\uparrow \neq g(P)_\downarrow{}^\uparrow$.

Weak morphisms behave better than pseudomorphisms with respect to composition. In fact, one can show easily:

COMPOSITION. *The class of weak morphisms is closed under composition. Similarly, for a weak morphism* $f : P \longrightarrow Q$ *and a pseudomorphism* $h : Q \longrightarrow R$ *the composite map* hf *is a pseudomorphism.*

Moreover, pseudomorphisms and weak morphisms are related as follows:

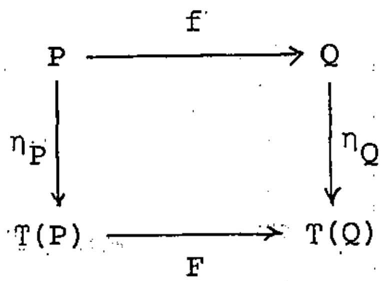
PROPOSITION. *Let* f *be a map between posets* P *and* Q *. Then*
(1) f *is Scott continuous iff* $\eta_Q f$ *is Scott continuous.*
(2) f *is a weak Galois map iff* $\eta_Q f$ *is a pseudo - Galois map.*
Hence f *is a weak morphism iff* $\eta_Q f$ *is a pseudomorphism.*



With this Proposition in hand, it is not hard to prove:

THEOREM 3. *Let* f *be a monotone map between up-complete posets* P *and* Q.
(1) f *is Scott continuous iff* $T(f)$ *preserves suprema.*
(2) If $T(f)$ *preserves infima then* f *is a weak Galois map. Conversely, if* f *is a weak Galois map and* Q *is a continuous poset then* $T(f)$ *preserves infima.*

COROLLARY. *For a map* $f : P \longrightarrow Q$ *where* P *is up-complete and* Q *is continuous, the following are equivalent:*
(a) f *is a weak morphism.*
(b) $T(f)$ *is a complete homomorphism.*
(c) There is a (unique) complete homomorphism F *such that the following diagram commutes:*



If P *is also continuous then each of these conditions is necessary and sufficient for* f *to be a* (CP-)*morphism. Hence the functor* T *induces an equivalence between the categories* CP *and* $\text{CD}_{\text{Cospec}}$.

Here $\underline{CD}_{Cospec}$ denotes the category of completely distributive lattices together with those complete homomorphisms which preserve cospectra (cf. [Kah, 1.8]).

The continuity assumption in Part (2) of Theorem 3 can be dropped whenever f is a Galois map. More precisely, we can say that the functor T preserves adjoints.

CONTRAPOSITION. *If* $f : P \longrightarrow Q$ *has a lower (resp. upper) adjoint* $d : Q \longrightarrow P$, *then* T(d) *is the lower (resp. upper) adjoint of* T(f).

Indeed, if f has a lower adjoint d then by the Trilemma f is quasiclosed, and consequently

$$T(f)(A) = \overline{f(A)} = \downarrow f(A) \quad \text{for all } A \in T(P).$$

From this equation, it follows at once that T(d) is the lower adjoint of T(f).

The Dilemma can now be restated in a more informative version. Theorem 2, Theorem 3 and the Proposition have the following

EFFECT. *Let* $\underline{C}$ *be any category of posets which has* $\underline{CD}$ *as a subcategory. If the forgetful functor from* $\underline{CD}$ *to* $\underline{C}$ *has a left adjoint with front adjunction* $\eta_P : P \longrightarrow T(P)$, $x \longmapsto \downarrow x$, *then* $\underline{C}$ *must be a subcategory of* $\underline{CL}$.

Conversely, from Theorem 1 we know that any full subcategory of $\underline{CL}$ has this universal property.

Finally, we would like to emphasize that most of the preceding assertions remain valid if the system of all directed subsets of P is replaced by an arbitrary "subset system" $\mathfrak{Z}(P)$ such that $\mathfrak{Z}$ - sets are preserved under isotone maps. In this general setting, the "co-selection"

$$\mathfrak{X}(P) = \{ A = \downarrow A : Z \in \mathfrak{Z}(P) \text{ and } Z \subseteq A \text{ implies } \bigvee Z \in A \}$$

plays the rôle of T(P). This approach leads to a very general theory which covers almost all known adjunctions, equivalences and dualities for posets. For example, one may take for $\mathfrak{Z}(P)$ the system of all singletons. For this special choice, $\mathfrak{X}(P)$ is the collection of all lower sets, and one obtains most of the results derived in Section 2 of [Hom]. On the other hand, the results of Section 4 are obtained for $\mathfrak{X}(P) = T(P)$.