SEMINAR ON CONTINUITY OF SEMILATTICES

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		7	28	82

Topic: Two remarkable continuous posets and an appendix to "The CL-compactification and the injective hull of a continuous poset"

REFERENCES:

Ba Banaschewski, B.: Essential extensions of T_0-spaces. GTA 7 (1977)

E Erné, M.: LNM 871, pp. 45-60

H_1 Hoffmann, R.-E.: Projective sober spaces LNM 871

H_2 " Continuous posets, prime spectra of completely distributive complete lattices and Hausdorff compactifications LNM 871

H_3 " : The CL-compactification and the injective hull of a continuous poset preprint (in need of correction) + SCS memo

H_4 " : Continuous posets: MacNeille completion and injective hull (in need of revision) + SCS memo

HoM Hofmann, K.H. and M.W. Mislove: A continuous poset whose compactification is not a continuous poset. The square is the injective hull of a discontinuous CL-compact poset. SCS-memo 5/28/82, rev. 7/2/82

Ho Hofmann, K.H. Bernhardina (The essential hull revised) SCS-memo 6/8/82, rev. 7/2/82

L Lawson, J.D. A strict extension of previous results on essential extensions SCS-memo 7/5/82

I have announced to circulate in a preprint volume both H_3 and H_4. I will not. (I have been promised that the volume – containing the material sent to me after the second Bremen workshop – will be ready for mailing soon.) What happened?

K.H. Hofmann and M.W. Mislove discovered a serious error in H_1 3.14: "(Every continuous poset has an injective hull, in $\underline{T}_0$, in its Scott topology, but) there are non-continuous posets with the Scott topology sober which have an injective hull in $\underline{T}_0$". The relevant error is in Ba, corollary 2, p.240. The proof of this is reduced to the observation that it may be seen from the proof of corollary 1 which is indeed correct (for a more detailed analysis see below).

Unfortunately this error affects much of the wording of H_2 and, partly, also H_4 ("X has an injective hull" has to be replaced by "X has ${}^sX$, the sobrification space of X, projective-sober or, equivalently, ${}^sX$ is a continuous poset in its Scott topology"), but the results are not intrinsically affected.

However, a basic claim of H_3 is false:

The CL-compactification C of a continuous poset P need not be a continuous poset

(If $e : (P, \sigma_P) \hookrightarrow (L, \sigma_L)$ denotes the injective hull for the continuous poset P, the CL-compactification of P is defined to be the order-extension

$$P \hookrightarrow C$$

with C = closure of $e[P]$ in L with regard to the CL-topology of L, endowed with the partial order inherited from L.)

It is shown in H_3 that

$$(C, \sigma_L | C)$$

is a sober space with an injective hull (viz. $(C, \sigma_L|C) \hookrightarrow (L, \sigma_L)$).

Indeed, In HoM K.H. Hofmann and M.W. Mislove provide a continuous poset P whose CL-compactification C (carries the Scott topology $\sigma_C = \sigma_L | C$, but) fails to be a continuous poset.

In Ho a ~~continuous~~ non-continuous poset P is constructed ~~to~~gether with a sober topology τ such that

a) (P, τ) has an injective hull

b) (P, σ_P) fails to have an injective hull

The following continuous poset P answers a natural problem:

The CL-compactification C of P does not have an injective hull with regard to the Scott topology σ_C

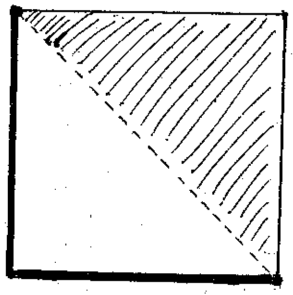
~~$P = \{(x,y) \in \mathbb{R}^2 \mid 0 \le$~~

$$P = \{(x,y) \in I^2 \mid x+y > 1\} \cup (\{0\} \times I) \cup (I \times \{0\})$$

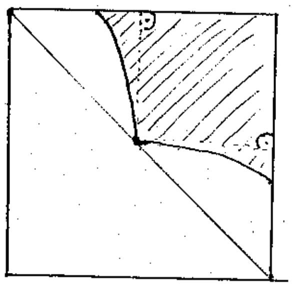
where I denotes the unit interval and P receives the natural order from I^2, then

$L = I^2$

$$C = P \cup \{(x,y) \in I^2 \mid x+y = 1\}$$



A σ_C-open neighborhood of (x,y) with $x+y=1$ $(x \neq 0,1)$ is



quite similar to K.H. Hofmann's example

in Ho and it results from the corrected Banaschewski criterion (cf. Ho or the included "appendix" 8.1(iii)) that (C, σ_C) fails to have an injective hull.

We construct a continuous poset P which has particularly remarkable properties. Let

$$L = \{a_n \mid n \in N\} \cup \{b_n \mid n \in N\} \cup \{a_0\} \cup \{c_n \mid n \in N\} \cup \{d, 0, 1\}.$$

with $N = \{1, 2, 3, \dots\}$ be partially ordered by

$a_0 < b_n$ (all n)

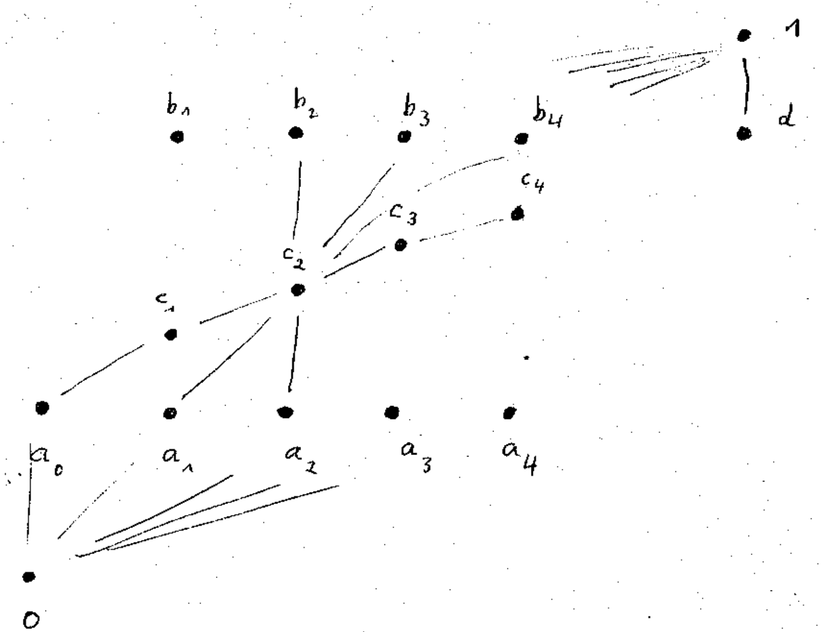
$a_k < c_\ell < b_m$ iff $k \leq \ell \leq m$, $\ell, m \geq 1$

$c_m \leq c_n$ iff $m \leq n$

$a_n < c_n < d$

(for natural numbers k, ℓ, m, n) such that $0, 1$ are the smallest and the greatest element of L, respectively.

This is a continuous lattice, indeed an algebraic lattice.



The only non-empty up-directed subset which does not contain its supremum is a cofinal subset of

$$\downarrow c_1 \cup \downarrow c_2 \cup \ldots$$

and has d as a supremum.

All elements of L are compact except for d.

$$\{c_n\} = \uparrow c_n - (\uparrow b_n \cup \uparrow c_{n+1})$$

Thus c_n is isolated in the CL-topology of L.

A basic CL-open neighborhood

of d in L is (w.l.o.g.) of the form

$$\uparrow c_n - (\uparrow b_{k_1} \cup \ldots \cup \uparrow b_{k_j})$$

for natural numbers $n, k_1, \ldots, k_j$ (finitely many).

A basic CL-open neighborhood of 0 is (w.l.o.g.) of the form

$$L - (\uparrow a_{k_1} \cup \ldots \cup \uparrow a_{k_n}).$$

~~Thus the CL compactification of the subposet~~

We consider the subposet P of L:

$$P = \{a_n \mid n \in \mathbb{N} \cup \{0\}\} \cup \{b_n \mid n \in \mathbb{N} \cup \{0\}\}$$

Clearly, P satisfies the a.c.c. (≡ ascending chain condition), hence is a continuous poset.

$$(P, \leq_P) \hookrightarrow (L, \leq_L)$$

is an injective hull by H_3, since

1. $P \hookrightarrow L$ preserves non-empty up-directed suprema
2. $P \hookrightarrow L$ preserves the way below

relation ($x \ll y$ in P iff $x \leq y$)

3. P generates L:

$$c_n = b_n \wedge b_{n+1}$$

$$0 = a_0 \wedge a_1$$

$$1 = \inf \emptyset$$

and d is the supremum of the chain

$$(c_n)_{n \in \mathbb{N}}$$

4. P is join-dense in L:

$$c_n = a_n \vee a_{n-1}$$

$$d = \sup \{c_n \mid n \in \mathbb{N}\}$$

$$1 = \sup \{b_n \mid n \in \mathbb{N}\}$$

$$0 = \sup \emptyset$$

The CL-compactification of P, i.e. the closure of P in L with regard to the CL-topology of L

is

$$C = P \cup \{0, d\}.$$

Note that

1. C has the a.c.c., hence (C, σ_C) has an injective hull

2. $$\sigma_L | C \neq \sigma_C$$

Every σ_L-neighborhood of $d \in L$ contains some $\uparrow c_n$, hence it contains $\{b_n, b_{n+1}, \dots\}$ for some n. Thus the σ_C-open set $\{d\}$ (since d is compact in C) is not the trace of ~~of~~ any σ_L-open set.

Thus a (continuous) poset with a.c.c. may carry a compatible sober topology different from the Scott topology which has an injective hull. By 8.5 (enclosed appendix) this phenomenon cannot arise for continuous lattices.

However it may be noted that although

$$\sigma_L | C \neq \sigma_C$$

in general, $\sigma_L | C$ is an <u>intrinsic</u> topology of C, the <u>CL</u>-compactification of the continuous poset P, since – as observed in H_3 –

$$C \hookrightarrow L$$

is the MacNeille completion of C.
(By the way, the MacNeille completion $M = L - \{d\}$ of P, fails to be a continuous lattice; related examples are discussed in E.)

Enclosed find an appendix to H_3 which is a revised version of a draft which I wrote before I had received the memo H_0 of K.H. Hofmann who also proved 8.1 (iii). 8.1 (ii) can be also deduced from a more recent result of J. D. Lawson L. (The references refer to the bibliography of H_3.)

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§ 8 Appendix

Correcting a mistake in $[Ba_2]$ cor.2, p.24o, we provide necessary and sufficient conditions in order that the greatest essential extension space λX of a T_o-space X be an injective T_o-space: Counterexamples to the claim made in $[Ba_2]$ were recently obtained by K.H.Hofmann and M.W.Mislove $[HM_2]$. We take $[Ba_2]$ section 2 for granted, but <u>no</u> information from $[Ba_2]$ section 3 will be used. (See $[Ho_2]$ for a somewhat different approach.)

Also, some additional comments are given correcting the statements of the results in $[H_6]$ and $[H_9]$ which are based upon $[Ba_2]$ cor.2, p.24o.

8.o For a T_o-space X, λX is – by the very construction – stable in ΦX under the formation of arbitrary joins (=suprema). Thus there is a "kernel operator" $k:\Phi X \to \lambda X$ assigning to every open filter F of x the greatest join filter

$$\bigvee \{\underline{O}(x) \mid x \in X, \underline{O}(x) \leq F\}$$

contained in F. This map k is left inverse to the embedding $\lambda X \hookrightarrow \Phi X$. (Indeed, by $[Ba_2]$ prop.3, p.239, $k:\Phi X \to \lambda X$ is the only continuous left inverse of the embedding $\lambda X \hookrightarrow \Phi X$ if there exists any.)

Note that

$$\bigvee \{\underline{O}(x) \mid x \in S\} = \{V \in \underline{O}(X) \mid \text{there are } x_1,\dots,x_n \in S\ (n \geq 0) \text{ and open neighborhoods } U_1,\dots,U_n \text{ of } x_1,\dots,x_n \text{ respectively with } U_1 \cap \dots \cap U_n \leq V\}$$

for every subset S of X.

8.1 <u>THEOREM</u>:
For a T_o-space X, the following are equivalent:
(i) The essential hull λX of X is an injective T_o-space.
(ii) There is a (topological) embedding $e:X \hookrightarrow J$ into an injective T_o-space J which is join-dense with regard to the specialization of partial order of J.

(iii) For every $x \in X$ and every open neighborhood V of x in X there exists an open neighborhood W of x in X, finitely many elements $y_1, y_2, \ldots, y_n$ $(n \geq o)$ of X and open neighborhoods $U_1, U_2, \ldots, U_n$ of $y_1, \ldots, y_n$, respectively, such that

$$W \subseteq \{z \in X \mid y_i \in \mathrm{cl}\{z\}\}$$

for every $i=1,\ldots,n$, and

$$U_1 \cap \ldots \cap U_n \subseteq V .$$

<u>Proof:</u>

(i) implies (ii): Evidently, $\lambda_X : X \hookrightarrow \lambda X$ is - by the very construction - join-dense with regard to specialization order (which coincides with the inclusion relation of λX and ΦX, respectively).

(ii) implies (iii): By Scott's result, $[Sc_2]$ 2.12 ($[C]$ II-3.8), J is a continuous lattice L endowed with its Scott topology σ_L. The sets

$$\Uparrow q = \{p \in L \mid q \ll p\} \quad (q \in L)$$

form an open basis of σ_L ($[C]$ II-1.1o(i)). We may clearly restrict ourselves to the basic open subsets of X,

$$V = X \cap \Uparrow q$$

with q ranging through L.

Suppose $x \in V = X \cap \Uparrow q$ for some $q \in L$. By the interpolation property of $\ll$ in a continuous lattice (C I-1.18), there is some $p \in L$ with $q \ll p \ll x$ in L, hence

$$x \in W := X \cap \Uparrow p \subseteq V .$$

Since, by hypothesis, $e : X \hookrightarrow J$ is join-dense, we have

$$p = \sup\{s \in X \mid s \leq p\}.$$

On the other hand,

$$y = \sup\{t \in L \mid t \ll y\}$$

for every $y \in L$ (since L is a continuous lattice). Consequently (by the associativity law for the operation "sup"),

$$p = \sup\{t \in L \mid t \ll y \leq p \quad \text{for some } y \in X\}.$$

Since $q \ll p$, it results that there are finitely many $t_1, \ldots, t_n \in L$ $(n \geq o)$ and $y_1, \ldots, y_n \in X$ with

$$q \leq \sup\{t_1, \ldots, t_n\}$$

and

$$t_i \ll y_i \leq p$$

for $i=1,\ldots,n$. It results that every neighborhood, in X, of y_i contains $W = X \cap \Uparrow p$, and there are open (in X) neighborhoods $U_i = X \cap \Uparrow t_i$ of y_i $(i=1,\ldots,n)$ with

$$U_1 \cap \ldots \cap U_n \subseteq V = X \cap \Uparrow q .$$

(iii) implies (i): We shall prove that the kernel operator $k : \Phi X \to \lambda X$ is a continuous map, hence a retraction in $\underline{T}_o$. Since ΦX is an injective T_o-space, then so is its retract λX.

Suppose F is any open filter of X and

$$k(F) \subseteq \Phi_V .$$

Then there are $x_1, \ldots, x_m$ $(m \geq o)$ and open neighborhoods $V_1, \ldots, V_m$ of $x_1, \ldots, x_m$ respectively with

$$\underline{O}(x_i) \subseteq F$$

for every $i=1,\ldots,m$ and

$$V_1 \cap \ldots \cap V_m \subseteq V .$$

By (iii), for every $i=1,\ldots,m$ there is an open neighborhood W_i of x_i and finitely many elements $y_i^1, \ldots, y_i^{n(i)}$ and open neighborhoods $U_i^1, \ldots, U_i^{n(i)}$ of $y_i^1, \ldots, y_i^{n(i)}$ respectively with

$$W_i \subseteq \{z \in X \mid y_i^j \in \mathrm{cl}\{z\}\}$$

or, equivalently,

$$\underline{O}(y_i^j) \subseteq W_i^{\varphi}$$

(where $W^{\varphi} = \{M \in \underline{O}(X) \mid W \subseteq M\}$ denotes the smallest member of Φ_W, the open filter generated by W)

for every $j=1,\ldots,n(i)$, and

$$U_i^1 \cap \ldots \cap U_i^{n(i)} \subseteq V_i .$$

It results that

$$\underline{O}(y_i^j) \subseteq (W_1 \cap \ldots \cap W_m)^{\varphi}$$

for every $i=1,\ldots,m$ and every $j=1,\ldots,n(i)$, and

$$\bigcap\{U_i^j \mid i=1,\ldots,m \text{ and } j=1,\ldots,n(i)\}$$
$$\subseteq \quad V_1 \cap \ldots \cap V_m \subseteq V .$$

Thus

$$k(W^{\varphi}) = \bigvee\{\underline{O}(y) \mid y \in X,\ \underline{O}(y) \subseteq W^{\varphi}\}$$

for $W := W_1 \cap \ldots \cap W_n$ contains V. Consequently, (because k is isotone and Φ_V is an upper set,) we have

$$k(G) \in \Phi_V$$

for every $G \in \Phi_W$.
Since

$$U_i^j \in \underline{O}(y_i^j) \subseteq W_i^\varphi$$

$$= W_1^\varphi \vee \ldots \vee W_m^\varphi \subseteq \underline{O}(x_1) \vee \ldots \vee \underline{O}(x_m)$$

$$\subseteq F,$$

we can also infer that $k(F) \in \Phi_V$, hence $k: \Phi X \to \lambda X$ is continuous (at F).

This completes the proof.

8.2 REMARKS:

i) Note that in 8.1(iii) necessarily

$$W \subseteq V.$$

ii) Suppose $e: X \hookrightarrow J$ is a join-dense topological embedding into an injective T_o-space $J=(L,\sigma_L)$. Let L' be the continuous lattice generated by $e[X]$ in J (in the sense that it is the smallest subset of J containing $e[X]$ closed under arbitrary infima and suprema of non-empty up-directed subsets). Then the induced map

$$e': X \hookrightarrow J' := (L', \sigma_{L'})$$

is the injective hull of X. (The arguments given in section 1 go through.)

8.3 DEFINITION:

Suppose X is a T_o-space with an injective hull $X \hookrightarrow \lambda X$. We say that

$$\deg X \leq r,$$

i.e. X has degree at most r (a natural number $\geq o$) iff 8.1(iii) can be fulfilled for every point x in X and every open neighborhood V of x in X by some $n \leq r$.

8.4 REMARK:

A T_o-space X with an injective hull satisfies $\deg(X) \leq 1$ iff for every $x \in X$ and every open neighborhood V of x there is some open neighborhood W of x and some $y \in V$ with

$$W \subseteq \{z \in X \mid y \in cl\{z\}\}.$$

B.Banaschewski ([Ba$_2$] cor.2, p.24o) observes that this class of T_o-spaces has an injective hull in $\underline{T}_o$, and he claims the other implication to be true, too. The error is hidden in the proof of [Ba$_2$] cor.1, p.239 (line 3 from below)

$$\underline{O}(x) = \bigvee k(\underline{F}\{U\})$$

need not be a set-theoretic union if λX is injective (but this is true if every join filter of X is a neighborhood filter, as it is assumed there).

In [H$_6$] 3.14 it is established that the continuous posets in their Scott topology are precisely those sober spaces X with an injective hull satisfying $\deg(X) \leq 1$. All the statements in [H$_9$] on spaces X with an injective hull (except for 4.3) require the additional hypothesis $\deg(X) \leq 1$. In this regard, the following is certainly of interest.

8.5 PROPOSITION:

Suppose a T_o-space X is a conditional o,v-semilattice with regard to its specialization order. If X has an injective hull, then $\deg(X) \leq 1$.

Proof:

A poset is a conditional o,v-semilattice if every finite subset which has an upper bound has a supremum.
In 8.1(iii) one may put

$$y = \sup\{y_1, \ldots, y_n\}.$$

where the "sup" is taken in $(X, \leq)$. Then

$$y \in U_1 \cap \ldots \cap U_n \subseteq V,$$

and

$$W \subseteq \uparrow y.$$

8.6 <u>COROLLARY</u>:

A T_o-space X is injective iff

i) X is sober,

ii) X has an injective hull in $\underline{T}_o$, and

iii) X is a o,v-semilattice in its specialization order.

<u>Proof</u>:

See $[H_9]$ 2.8.

8.7 <u>COROLLARY</u>:

If a T_1-space X has an injective hull, then X is discrete.

<u>Proof</u>:

Suppose X has at least two points. For $x \in X$ choose some neighborhood $V \neq X$ of x. Then let $W \in \underline{O}(X)$ and $y_1,\dots,y_n \in X$ $(n \geq o)$ and $U_1,\dots,U_n$ be chosen as in 8.1(iii)

Since every point-closure in a T_1-space is a singleton,

$$x \in W \subseteq \{z \in X \mid y_i \in cl\{z\}\}$$

implies - if $n \neq o$ - that $y_1=\dots=y_n=x$, hence $W=\{x\}$ is open. If n=o, then

$$x = U_1 \cap \dots \cap U_n \subseteq V$$

contradicting the hypothesis that $X \neq V$.

8.8 <u>COROLLARY</u>:

Suppose A is a closed subspace of a T_o-space X. If X has an injective hull in $\underline{T}_o$, then so has A.

<u>Proof</u>:

In order to verify 8.1(iii) let $x \in V' \in \underline{O}(A)$. Then $V'=V \cap A$ for some $V \in \underline{O}(X)$, and we may choose $W \in \underline{O}(x)$, some points $y_1,y_2,\dots,y_n$ $(n \geq o)$ in X and open neighborhoods $U_1,\dots,U_n$ (in X) of $y_1,\dots,y_n$ respectively satisfying 8.1(iii). The requirement

$$x \in W \subseteq \{z \in X \mid y_i \in cl\{z\}\}$$

(i=1,...,n) guarantees

$$y_i \in cl\{x\} \subseteq A$$

so that we may use $W'=W \cap A$ and $U_i':=U_i \cap A$ in order to fulfill 8.1(iii) for A instead of X.

8.9 <u>PROPOSITION</u>:

Suppose $(X_i)_{i \in I}$ is a family of T_o-spaces which have an injective hull in $\underline{T}_o$. Then $\prod_{i \in I} X_i$ has an injective hull provided that

$$K(I)=\{i \in I \mid X_i \text{ does not have a smallest element in its specialization order}\}$$

is finite.

<u>Proof</u>:

(1) First note that if X and Y have an injective hull, then so has $X \times Y$ (use 8.1(iii)).

(2) Suppose now $K(I)=\emptyset$ and let o_i denote the smallest element of X_i in its specialization order. By 8.1(ii), there are injective T_o-spaces J_i and join-dense (topological) embeddings $X_i \hookrightarrow J_i$. Clearly, $\prod_I J_i$ is injective. Let

$$a_i \in J_i \quad (i \in I),$$

then, by hypothesis,

$$a_i = \sup A_i$$

for some subset A_i of X_i. We may assume that $o_i \in A_i$, hence $A_i \neq \emptyset$. Then

$$(a_i)_{i \in I}=(\sup A_i)_{i \in I}=\sup(\prod_{i \in I} A_i).$$

This proves that $\prod_{i \in I} X_i$ is join-dense in $\prod_{i \in I} J_i$, hence it has an injective hull in $\underline{T}_o$ by 8.1(ii).

Combining (1) and (2), we establish the assertion.

A product of discrete spaces may fail to be discrete, but it is always T_1. Thus (by 8.7) the class of all T_o-spaces with an injective hull in $\underline{T}_o$ fails to be productive.

8.1o REMARKS:

The non-validity of one implication of [Ba_2] cor.2, p.24o makes several results questionable which were based on this claim, e.g.: Is every T_D-space ($=T_{1/2}$-space, [Br] II p.7; "points are locally closed") with an injective hull in $\underline{T}_o$ Alexandrov-discrete? (Cf. [H_5] 4.3.)

K.H.Hofmann observes that the class of T_o-spaces with an injective hull is not open-hereditary (disproving [Ba_2] cor.4, p.24o).

8.11 REMARK:

The requirement to have an injective hull in $\underline{T}_o$ does not impose any restriction on the specialization partial order: For every poset P, (P, α_P) has an injective hull in $\underline{T}_o$ ([H_5] 4.2).

However, a sober space X with an injective hull in $\underline{T}_o$ yields always an "almost-continuous" poset $(|X|, \leq_X)$ in the specialization order $\leq_X$ in the sense that it is up-complete (by sobriety, cf. [Wy]) and, for every $x \in X$,

$$x = \sup\{y \in X \mid y \ll x\}$$

where $\ll$ denotes the way below relation of $(X, \leq_X)$. (The latter assertion results from the fact that (1) X is join-dense in λX, by 1.o(i), and (2) every element F of λX is, by injectivity of λX, a supremum of elements way below F in λX, since the embedding $\lambda_X : X \hookrightarrow \lambda X$ is an order-embedding preserving suprema of non-empty up-directed subsets, hence reflecting the way below relation, by 1.2(b)). Note however that the set

$$\{y \in X \mid y \ll x\}$$

need not be up-directed, as K.H.Hofmann and M.W.Mislove [HM_2] demonstrate. K.H.Hofmann [Ho_2] observes that the topology of X need not be the Scott topology (this may even fail to have an injective hull).

8.12 REMARK:

The notion of a degree for injective hulls leads to a natural (new) dimension function i-dim for continuous lattices L themselves ("injectivity dimension"): i-dimL is at least n ($n \geq o$) iff (L, σ_L) is the injective hull of a sober space X of degree at least n.

The unit interval I has i-dimension 1. The example provided by K.H.Hofmann and M.W.Mislove [HM_2] shows that i-dim$I^2 \geq 2$ and, analogously, i-dim$I^n \geq n$. Is it true that i-dim$I^n = n$? Are there continuous lattices L with i-dimL$= \infty$?

8.13 PROBLEMS:

~~Is there a continuous poset P which carries a sober topology~~ $\cancel{\tau \neq \sigma_P}$ ~~inducing the given partial order such that~~ $\cancel{(P,\tau)}$ ~~has an injective hull?~~ Does every almost-continuous poset carry a ~~(unique?)~~ sober topology inducing the order and having an injective hull?

One easily sees that for a given poset P the supremum of every non-empty family of compatible topologies with an injective hull also has an injective hull. Is there always a coarsest compatible topology on a poset which has an injective hull (yielding the empty-indexed supremum)? The finest such topology is the Alexandrov-discrete topology ([H_5] 4.3).