SEMINAR ON CONTINUITY IN SEMILATTICES

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TOPIC: Algebraic posets and compactly generated posets

REFERENCES:

[BF] G.Birkhoff and O.Frink, Representations of lattices by sets. Trans AMS 64 (1948), 299-316

[C] A Compendium

[CD] P.Crawley and R.P.Dilworth, Algebraic theory of lattices. Prentice Hall, Englewood Cliffs 1973

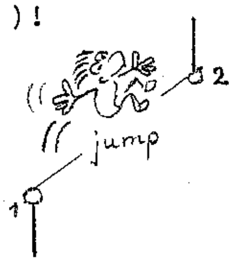
[Di] K.-H. Diener, Über zwei Birkhoff-Frinksche Struktursätze der allgemeinen Algebra. Arch.Math. 7 (1956), 339-346

[Reh] R.-E. Hoffmann, Continuous posets and adjoint sequences. Semigroup Forum 18 (1979), 173-188

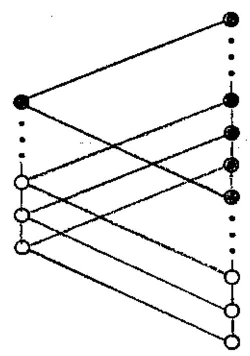
A poset P is said to be *weakly atomic* if every proper interval [x,y] of P contains a jump, i.e. a pair of elements $u < v$ such that $|[u,v]| = 2$. It has been observed by Birkhoff and Frink [BF] that every algebraic complete lattice is weakly atomic. However, the initial proof given in [BF] contained an error which was corrected by K.-H. Diener [Di]. Probably the most simple proof is the following. Any proper interval [x,y] of an algebraic lattice is again an algebraic lattice (with respect to the induced order). Hence, as $x < y$, there exists a compact element v of [x,y] with $x < v$ (which need not be compact in L !), and by definition of compactness, the nonempty half-open interval [x,v[is up-complete and has therefore a maximal element u. Thus [u,v] is the required jump.

Recall that an up-complete poset P is called *algebraic* if every element of P is the supremum of a directed set of compact elements [Reh]. Now it is an obvious question whether the above conclusion may be extended to algebraic posets. Unfortunately, there is an

OBSTACLE. An interval of an algebraic poset need not be algebraic (not even continuous, in contrast to [C; Ch.I, Ex.1.28])!



COUNTEREXAMPLE 1 .



Of course, this algebraic poset is weakly atomic, but it contains a complete interval which is not even continuous.

FACT . <u>An algebraic poset need not be weakly atomic</u>.

COUNTEREXAMPLE 2 . For $a,b \in \mathbb{R}$, set

$$[a,b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \ , \ [a,b[\ = \{ x \in \mathbb{R} : a \leq x < b \} ,$$

$$L = \{ [0,a] : a \in [0,1[\ \} \cup \{ [0,a[\ : a \in [0,1] \} .$$

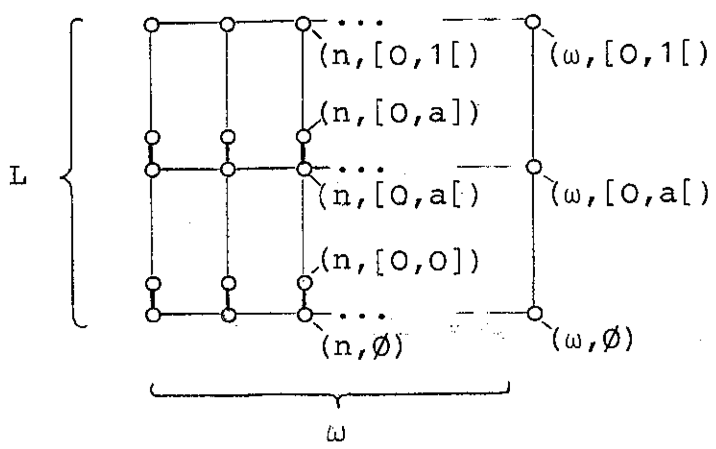
Then L is a complete chain with respect to inclusion, consisting of all lower ends of the interval $[0,1[$. Hence L is an algebraic lattice, and the compact elements of L are the intervals $[0,a]$ with $a \in [0,1[$ and the empty interval $\emptyset = [0,0[$.

Another algebraic chain is $\omega + 1 = \omega \cup \{\omega\}$, where ω is the chain of all natural numbers (the compact elements of $\omega + 1$). Consider

$$P = (\omega \times L) \cup \{ (\omega, [0,a[) : a \in [0,1] \} ,$$

together with the partial order

$$(m,I) \leq (n,J) \iff m \leq n \ , \ I \subseteq J \ , \text{ and } \ I = J = [0,a] \text{ implies } m = n \ .$$



It is not hard to verify (i.e., I hope that I did not fail in showing) that P is an algebraic poset, the compact elements being the pairs (n,I) with $n \in \omega$ and $I = \emptyset$ or $I = [0,a]$ for some $a \in [0,1[$. But the interval $\uparrow(\omega,\emptyset) = \{ (\omega,[0,a[) : a \in [0,1] \} \subseteq P$ is isomorphic to the unit interval $[0,1]$ and contains no jump.

In spite of these counterexamples, a slight generalization of the fact that algebraic lattices are weakly atomic is possible. Call a poset P *chain-complete* (or *Dedekind-complete*) if every nonempty chain of P has a supremum and an infimum (i.e., P and its dual are up-complete). P is *compactly generated* if every element of P is a supremum of compact elements.

PROPOSITION . Every compactly generated chain-complete poset is weakly atomic. In particular, this is true for every algebraic poset whose dual is up-complete.

PROOF. Let $[x,y]$ be an interval of a compactly generated chain-complete poset P with $x < y$, and let C be a maximal chain in $[x,y]$. Then C must be complete (and sups and infs agree with those formed in P ! For this conclusion, we need up- and down-completeness). Choose a compact element c of P with $c \leq y$ but $c \not\leq x$. Let $v = \inf\{ z \in C : c \leq z \}$. Then $c \leq v$ (!) and $x < v$. Further, $D = \{ z \in C : z < v \}$ is a chain with $c \not\leq z$ for all $z \in D$, whence, by compactness of c, $c \not\leq u = \sup D$, and so $u < v$. Thus $[u,v]$ is a jump in $[x,y]$, as desired.

It should be mentioned that the proofs of Diener [Di] and Crawley/Dilworth [CD, 2.2] for the weak atomicity of algebraic lattices involve the existence of certain finite (undirected) suprema and do not work in the present more general setting. Notice also that a compactly generated up-complete poset need not be algebraic.

COUNTEREXAMPLE 3 . Let X be an uncountable set, and let P denote the system of all subsets of X which have either at most one element or a countable complement. This system is closed under directed unions and countable intersections. In particular, it is an up-complete semilattice with respect to inclusion. The compact elements are those subsets which have at most one element. Hence P is compactly generated but not algebraic.