SEMINAR ON CONTINUITY IN SEMILATTICES

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TOPIC: Meet-continuous lattices in which meet is not continuous

REFERENCES: [Bi] G.Birkhoff, Lattice Theory

[C] A Compendium of CL

[E 1] M.Erné, Order-topological lattices. Glasgow Math.J. 21 (1980) 57 - 68

[E 2] M.Erné, Topologies on products of partially ordered sets III: Order convergence and order topology. Algebra Universalis 13 (1981) 1 - 23

[Fla] J.Flachsmeyer, Einige topologische Fragen in der Theorie der Booleschen Algebren. Arch.Math. 16 (1965) 25 - 33

[Flo] E.E.Floyd, Boolean algebras with pathological order topologies. Pacific J.Math. 5 (1955) 687 - 689.

At the Bremen Workshop 1982 on Continuous Lattices, the problem arose to find examples of meet-continuous lattices in which the binary meet-operation is not (jointly) continuous with respect to the Scott topology. As complete Boolean lattices are always join- and meet-continuous, it appears reasonable to look for examples within this class. It turns out that a complete Boolean lattice whose meet-operation is continuous in the Scott topology (or order topology) must be Hausdorff in its order topology (see the Theorem below). Papers of Floyd [Flo] and Flachsmeyer [Fla] provide us with enough examples of Boolean lattices whose order topology fails to be Hausdorff, e.g. the lattice of regular open subsets of $\mathbb{R}$. Such lattices cannot be topological (semi)lattices with respect to the order topology $\mathcal{O}(B)$ or the Scott topology $\sigma(B)$; in particular,

$$\mathcal{O}(B \times B) \neq \mathcal{O}(B) \times \mathcal{O}(B) \quad \text{and} \quad \sigma(B \times B) \neq \sigma(B) \times \sigma(B) .$$

THEOREM. *Consider the following statements on a complete Boolean lattice* B *:*

(a) B *is atomic (i.e. isomorphic to a power set).*

(b) B *is continuous.*

(c) $\mathcal{O}(B \times B) = \mathcal{O}(B) \times \mathcal{O}(B)$.

(d) $\sigma(B \times B) = \sigma(B) \times \sigma(B)$.

(e) $(B, \mathcal{O}(B))$ *is a topological lattice* ($\vee$*-semilattice*, $\wedge$*-semilattice).*

(f) $(B, \sigma(B))$ *is a topological lattice.*

(g) $(B, \sigma(B))$ *is a topological* $\wedge$*-semilattice.*

(h) $(B, \sigma(B))$ *is a topological* $\vee$*-semilattice.*

(i) The Bi-Scott topology $\sigma(B) \vee \sigma(B^{op})$ *is Hausdorff.*

(j) The order topology $\mathcal{O}(B)$ *is Hausdorff.*

(k) The Scott topology $\sigma(B)$ *is sober.*

The following implications are always true:

$$\begin{array}{ccccccccc} (a) & \Leftrightarrow & (b) & \Rightarrow & (c) & \Rightarrow & (d) & & & & \\ & & & & \Downarrow & & \Downarrow & & & & \\ & & & & (e) & \Rightarrow & (f) & \Rightarrow & (g) & \Rightarrow & (i) & \Rightarrow & (j) \\ & & & & & & \Downarrow & & & & \\ & & & & & & (h) & \Rightarrow & (k) & . \end{array}$$

All implications except (g) $\Rightarrow$ (i) are easily verified or well-known from [C], [E 1] and [E 2] . The implication (g) $\Rightarrow$ (i) may be strengthened as follows. *If* $(B, \sigma(B))$ *is a topological* $\wedge$*-semilattice then for* $x \nleq y$ *in* B *there exist* $U \in \sigma(B)$ *and* $V \in \sigma(B^{op})$ *with* $x \in U$, $y \in V$ *and* $U \cap V = \emptyset$. *In other words, the relation* $\leq$ *is closed with respect to the order topology, and in particular,* $\mathcal{O}(B)$ *is Hausdorff.*

PROOF. $x \nleq y$ implies $x \in B \setminus {\downarrow}y \in \sigma(B)$. As $x = x \wedge x$, there exists $U \in \sigma(B)$ with $x \in U$ and $U \wedge U \subseteq B \setminus {\downarrow}y$, i.e. $u \wedge v \nleq y$ for all $u,v \in U$. Define $V := \{ v \in B : u \wedge v \leq y$ for some $u \in U \}$. Then $y \in V$, $U \cap V = \emptyset$, and $V \in \sigma(B^{op})$. Indeed, if F is a filter in B with $\bigwedge F \in V$ then $u \wedge \bigwedge F \leq y$ for some $u \in U$, whence $u \leq y \vee (\bigwedge F)' = \bigvee \{ y \vee z' : z \in F \}$. This is a directed sup, and since $u \in U \in \sigma(B)$, it follows that $y \vee z' \in U$ for some $z \in F$, i.e. $w \wedge z \leq y$ for some $w \in U$, and so $z \in F \cap V$.

Certainly it would be interesting to investigate which of the implications in the Theorem may be inverted.