SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC: On the pseudo-spectrum of a continuous distributive lattice

REFERENCES: [C] Gierz,G., et al. , A Compendium of Continuous Lattices Springer- Verlag 1980

[H_1] Hoffmann, Rudolf -E. , The Fell compactification revisited Mathematik Arbeitspapiere der Universität Bremen 27 (1982),68-141

[H_2] - , The trace of the weak topology ... , SCS-Memo 1-9-83

[HL] Hofmann,K.H., and J.D.Lawson, The spectral theory of distributive continuous lattices, Trans.Amer.Math.Soc. 246 (1978),285-310

[HM] Hofmann,K.H., and M.Mislove, Local compactness and continuous lattices, Lecture Notes in Math.871 (1981), 209-248.

In his recent Memo 1-9-83, R.-E.Hoffmann refers to a private communication on my part which thereby ceased to be private. I had no particular desire to rush a memo about it; but in order to understand his memo fully,the SCS had better be informed on the content of my thoughts as they were presented in the communication to which Rudolf refers.

In his paper [H_1] , he makes some very interesting discoveries about the pseudo-spectrum of a continuous lattice; indeed I consider the theory of the pseudo-spectrum incomplete without them, and it amazes me that the ones among them,which I consider most important, were not made a long time ago. My comments have the purpose to show that these results can be derived from the basic theory by direct methods. I think that my presentation should be compared with [H_1]; perhaps such a comparison retroactively explains my desire to find a direct route to Hoffmann's discoveries.

The spectrum Spec L of a continuous distributive lattice L or, as I will say more succinctly, a <u>continuous</u> <u>frame</u> L, is the set of all primes with the hull kernel topology $\omega|$Spec L. (Recall that we have ceased to consider 1 as a prime.) The pseudo-spectrum ΨL of a continuous frameL is the set of all pseudo-primes, i.e. elements p with p = sup P for a prime ideal P. (After R.-E.Hoffmann, Essentially complete T_0-spaces II, Math.Z. 179 (1982),73-90, this concept has been elucidated,too: L itself is not a prime ideal; thus 1 may or may not be in ΨL.) Prior to [H_1] the pseudospectrum was not at all considered as endowed with a topology which extends that of the spectrum. One of the important points in [H_1] is, that such a topology exists, that it is quasicompact with a whole row of additional desirable properties. In this sense, ΨL will be a compactification of Spec L. In fact it was not even clear

clear to me at all that a locally quasicompact topological space should have any decent, let alone natural quasicompactification with good properties. Only much finer topologies had been considered on ψL before, namely, $\lambda|\psi L$, and to some extent this was simply a consequence of the fact that $\psi L = \mathrm{cl}_\lambda$ Spec L. (This was fully clarified by R.-E.Hoffman in his Math.Z.paper, loc.cit.) Thus this memo is concerned with a direct approach to the pseudo-spectral theory of continuous frames. The terminology is that of [C] and [HM].

1. The lower topology on Lawson closed subsets.

In this section, L *is a continuous lattice and* C *a* λ-*closed subset*.

1.1. LEMMA . $(C, \omega|C)$ is locally quasicompact and quasicompact T_o.
Proof. Since C is λ-compact and $\omega\subseteq\lambda$ clearly $(C,\omega|C)$ is quasicompact. Now let U be an ω-neighborhood of $c\in C$ in C. Then there are finitely many points $x_1,\ldots,x_n$ of L such that the set $V(x_1,\ldots,x_n) = C \setminus (\uparrow x_1\cup\ldots\cup\uparrow x_n)$ is an ω-neighborhood of c in C contained in U. (See [C], p.142; 1.1.)By the continuity of L we find points $y_k \ll x_k$ with $y_k \nleq c$, k= 1,...,n. Then $c \in V(y_1,\ldots,y_n) \subseteq C \setminus (\Uparrow y_1\cup\ldots\cup \Uparrow y_n) \subseteq V(x_1,\ldots,x_n) \subseteq U$. But then $C\setminus (\Uparrow y_1 \cup\ldots\cup \Uparrow y_n)$ is an ω-neighborhood of $c \in C$ which is contained in U and is λ-closed in C, hence λ-compact, and thus, a fortiori, ω-quasicompact. This shows that $(C,\omega|C)$ is locally quasicompact. Since ω is a T_o-topology, the Lemma is proved.#

1.2. LEMMA. Let Q be a saturated subset of $(C,\omega|C)$ (cf. [C] , p.258). Then $Q= C\cap\downarrow Q$, and the following statements are equivalent:

(1) Q is ω-quasicompact.

(2) $\downarrow Q$ is σ-closed.

(3) $\downarrow Q$ is λ-compact.

(4) Q is λ-compact.

Proof. By definition, Q is saturated iff $Q = C\cap \bigcap\{U\in\omega\colon Q\subseteq U\}$. If we set $S= \bigcap\{U\in\omega : Q\subseteq U\}$ then S us a lower set since all $U\in\omega$ are lower sets. The relation $Q\subseteq S$ then implies $\downarrow Q\subseteq S$. If $x\in L\setminus \downarrow Q$, then $U = L\setminus\uparrow x \in\omega$ with $Q \subseteq L\setminus\uparrow x = U$, but $x\notin U$. Hence $x \notin S$. Thus $\downarrow Q = S$ and $Q = C\cap\downarrow Q$.

The equivalence of (2) and (3) is clear since (L,λ) is compact and a lower set is σ-closed iff it is λ-closed (cf.[C],p. 144). The implication (3) $\Rightarrow$ (4) follows since C is λ-compact and $Q = C\cap\downarrow Q$.Trivially, (4) $\Rightarrow$ (1). It remains to show that (1) implies (2). Let $D\subseteq\downarrow Q$ be directed and set d = sup D. For $q\in D$ we have $\uparrow q\cap Q \neq \emptyset$; thus $\{\uparrow q\cap Q\colon q\in D\}$ is a filterbasis of ω-closed subsets of Q. If Q is ω-quasicompact, by (1) this filterbasis has a non-empty intersection $\bigcap\{\uparrow q\cap Q\colon q\in D\} = Q\cap\bigcap\{\uparrow q : q\in D\} = Q\cap\uparrow d$. Thus $d\in\downarrow Q$. This proves (2).#

1.3. LEMMA. $(C,\omega|C)$ is super-sober. (See [C],p.310; 1.10)
Proof. Let $\mathcal{F}$ be an ultrafilter on C. Then $x= \lim_\lambda\mathcal{F}$ exists since C is λ-compact.

Since $\omega \subseteq \lambda$, then x is also an ω-limit point of $\mathcal{F}$, and then all points of $\uparrow x$ are limit points of $\mathcal{F}$, since all ω-open sets are lower sets. Now let y be a limit point of $\mathcal{F}$. We claim $y \in \uparrow x_{\wedge}$ $= cl_{\omega}\{x\}$ and thereby finish the proof. Assume not. Then, by the continuity of L there is a $u \ll x$ with $y \notin \uparrow u$. Since $x = \lim_{\lambda} \mathcal{F}$, there is an $F \in \mathcal{F}$ with $F \subseteq \Uparrow u$. But $L \setminus \uparrow u$ is an ω-open neighborhood of the limit point y of $\mathcal{F}$, hence there is a $G \in \mathcal{F}$ with $G \subseteq L \setminus \uparrow u$. But then $F \cap G = \Uparrow u \cap (L \setminus \uparrow u) = \emptyset$ and this is a contradiction. #

1.4. REMARK. The set $Q(C, \omega|C)$ of saturated quasicompact subsets of $(C, \omega|C)$ is closed under arbitrary intersections (and, of course, finite unions). In the opposite order to that of containment, $Q(C, \omega|C)$ is a continuous frame.

Proof. By Lemmas 1.1, and 1,3, the assertion follows from [HM], p.238; 4.8. A simple direct proof that $Q(C, \omega|C)$ is closed under intersections follows from Lemma 1.3.#

Shall we summarize?

1.5. THEOREM. If a Lawson closed subset C of a continuous lattice is endowed with the topology $\omega|C$ induced from the lower topology, then the resulting space is locally quasicompact, quasicompact super sober. The set $Q(C, \omega|C)$ of saturated compact subsets is closed under intersections (and is in fact a frame in the order which is opposite to containment). Its patch topology is compact.#

(For a reminder of the patch topology see [C],261 or, better, [HM],p.236 and p.238.)

2. The pseudo-spectrum of a continuous frame.

2.1. DEFINITION. The pseudo-spectrum ΨL of a continuous frame L is the set of all pseudo-primes of L endowed with the topology $\omega|\Psi L$ induced from the lower topology.

2.2. THEOREM. The pseudo-spectrum ΨL of a continuous frame L is a quasicompact, locally quasicompact, super sober space containing the spectrum Spec L as a dense subspace. When equipped with the patch topology, $\Psi_p L$ is compact (T_2) and the subset $\mathrm{Spec}_p L$ is still dense.

Proof. Since ΨL is the λ-closure of Spec L, this all follows from Theorem 1.5.#

In [H_1], R.-E.H. proves that the set ΨL, somehow, carries a quasicompact, locally quasicompact, super sober topology extending that of Spec L (see [H_1], 4.11, p. 112). In [H_2] he shows that it is equal to $\omega|\Psi L$ by a self-contained argument.

In which way is Ψ a functor?

2.3. THEOREM. The assignment $L \mapsto \Psi L$ is a contravariant functor from the category CF of continuous frames and frame maps presering $\ll$ to the category QP of quasicompact, locally quasicompact, super sober spaces and perfect maps which associates with a CF -morphism $d: L_1 \longrightarrow L_2$ the function $g|\Psi L_2 : \Psi L_2 \longrightarrow \Psi L_1$, where $g: L_2 \longrightarrow L_1$ is the upper adjoint of d.

Proof. Since d is a frame map preserving the way below relation, g is a CL-map preserving spectra. (See [C], pp.180 and 188, or [HI]). Thus $g(\Psi L_2) = g(\mathrm{cl}_\lambda(\mathrm{Spec}\, L_2)) \subseteq \mathrm{cl}_\lambda\, g(\mathrm{Spec}\, L_2)$ (since g is λ-continuous) $\subseteq \mathrm{cl}_\lambda \mathrm{Spec}\, L_1$ (since g preserves spectra) $= \Psi L_1$. Thus $\Psi(d) = g|\Psi L_2 : \Psi L_2 \longrightarrow \Psi L_1$ is well defined. If $d_1 : L_1 \longrightarrow L_2$ and $d_2 : L_2 \longrightarrow L_3$ are CF -maps with upper adjoints g_1, g_2, respectively, then $g_1 g_2$ is the upper adjoint of $d_2 d_1$. Hence our definition yields a contravariant functor.

We recall that a map is called perfect iff it is continuous and the inverse image of a quasicompact saturated set is quasicompact. If Q is saturated quasicompact in ΨL_1, then $Q = \downarrow Q \cap \Psi L_1$ and $\downarrow Q$ is λ-closed by Lemma 1.2. The inverse image $\Psi(d)^{-1}(Q) = g^{-1}(Q) \cap \Psi L_2 = g^{-1}(\downarrow Q) \cap \Psi L_2$ is now quasicompact saturated in ΨL_2 by Lemma 1.2 since g is λ-continuous. Thus $\Psi(d) = g|\Psi L_2$ is perfect.#

This theorem is first proved by R.-E.Hoffmann in $[H_1]$, pp.124-133, Theorem 6.8.[1)]

If L is a continuous frame, then $0(\Psi L)$ is a continuous frame by 2.2 which has the additional properties that 1 $(=\Psi L)$ is compact and that $\ll$ is multiplicative. (See [HL], p.302, or [HM], 4.8.). If $\mathcal{L}$ is a continuous frame with compact identity and multiplicative way-below relation, then $\Psi\mathcal{L} = \mathrm{Spec}\,\mathcal{L}$ (cf. [HI], or R.-E.Hoffmann, Math.Z.loc.cit.). Hence $\Psi 0(\Psi L) = \mathrm{Spec}\; 0(\Psi L) = \Psi L$ under the standard isomorphisms since ΨL is sober. This shows

2.4.PROPOSITION. The functions $p \mapsto L \setminus \uparrow p : \Psi L \longrightarrow \Psi 0 \Psi L$ and $U \mapsto \inf L \setminus U : \Psi 0 \Psi L \longrightarrow \Psi L$ are mutually inverse homeomorphisms.#

The two continuous frames L and $\mathcal{L} = 0\Psi L$ are linked through a pair of adjoint maps $d : L \longrightarrow \mathcal{L}$, $d(x) = \Psi L \setminus \uparrow x$, and $g : \mathcal{L} \longrightarrow L$, $g(U) = \inf(\Psi L \setminus U)$. Indeed $d(x) \subseteq U$ iff $\Psi L \setminus U \subseteq \uparrow x$ iff $x \leq \inf(\Psi L \setminus U) = g(U)$. An element U is in Spec $\mathcal{L}$ if $U = L \setminus \{p\}^- = L \setminus \uparrow p$ for some $p \in \Psi L$. Since ΨL is inf-dense in L, we have $g(U) = p$. Thus g maps $\mathrm{Spec}\,\mathcal{L} = \Psi\mathcal{L}$ bijectively onto ΨL. If $x \ll y$ in L, then by the interpolation property ther is a $z \in L$ with $x \ll z \ll y$, whence $\Psi L \setminus \uparrow x \subseteq \Psi L \setminus \Uparrow z \subseteq L \setminus \uparrow y$. Now $\Psi L \setminus \Uparrow z$ is ω- quasicompact by Lemma 1.2, whence $d(x) = \Psi L \setminus \uparrow x \ll \Psi L \setminus \uparrow y = d(y)$. Thus d preserves $\ll$, whence g is σ-continuous, i.e. a CL-morphism. We have made the following remark:

2.5.PROPOSITION. The function $g_L : 0\Psi L \longrightarrow L$, $g_L(U) = \inf(\Psi L \setminus U)$, is a surjective CL-map between continuous frames which maps Spec $0\Psi L$ homeomorphically onto ΨL relative to both ω and λ. #

It lower adjoint d is rarely a frame map: Since g_L preserves spectra iff $\Psi L = \mathrm{Spec}\, L$, its lower adjoint d is a frame map iff it is an isomorphism.

The basic information contained in the theorems easily translates into general topology statements. If X is a locally quasicompact sober space, then ${}^{f}X = \Psi 0(X)$ is a quasicompact, locally quasicompact, super sober space into which X is densely embedded. The functor 0 from the category of locally quasicompact spaces and perfect maps goes contravariantly into the category CF, whence $X \longmapsto {}^{f}X$ is a functor from this category into its full subcategory QP.

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1) One must bear in mind Memo 1-9-83 in this context.

3. The pseudo-spectrum and duality.

In HL it was shown that in a continuous frame L the function $Q \mapsto L\setminus\downarrow Q : Q(\mathrm{Spec}\, L) \longrightarrow \mathrm{OFilt}\, L$ from the continuous semilattice $Q(\mathrm{Spec}\, L)$ of all quasicompact saturated sets of $\mathrm{Spec}\, L$ under ω onto the Lawson dual of L consisting of all Scott open filters was an isomorphism. With $\mathcal{L} = O(\psi L)$ we thus obtain an isomorphism $Q(\psi L) = \mathrm{OFilt}\,\mathcal{L}$, and since $Q(\psi L)$ is a frame by 1.1, in particular, the Lawson dual of $\mathcal{L}$ is a frame. R.-E.Hoffmann discovered a remarkable frame which serves as an isomorphic characterisation of $Q(\psi L)$.

3.1. <u>LEMMA.</u> For a filter U in a continuous frame L and a pseudo-prime $p \in \psi L$, the following conditions are equivalent:

(1) $p \in \sigma$-interior U .

(2) There is a $u \in U$ with $u \ll p$.

(3) $p \in \Uparrow U$.

(4) $\Downarrow p \cap U \neq \emptyset$.

(5) For all prime ideals P we have $p \leqslant \sup P \Rightarrow P \cap U \neq \emptyset$.

Proof. In any continuous lattice L, the sets $\Uparrow x$, $x \in L$ form a basis for σ . Hence (1) $\Rightarrow$ (2) . The equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are trivial. (2) $\Rightarrow$ (5): Let $u \in U$, and $u \ll p \leqslant \sup P$ for some prime ideal P. Then there is an $x \in P$ with $u \leqslant x$ by the definition of $\ll$. Hence $x \in U \cap P$. Not(4) $\Rightarrow$ Not(5): If the ideal $\Downarrow p$ and the filter U are disjoint, then in the frame L there is a prime ideal P with $\Downarrow p \subseteq P$ and $P \cap U = \emptyset$; evidently $p = \sup \Downarrow p \leqslant \sup P$.#

We now define $\pi(U) = \psi L \setminus \Uparrow U$. By Lemma 3.1, $\pi(U) = \psi L \cap \downarrow \pi(U)$ is λ-closed, hence is a member of $Q(\psi L)$ by Lemma 1.2. Since L is a frame, $U = \bigcap\{L \setminus P : P$ is a prime ideal of L with $U \cap P = \emptyset\} = L \setminus \bigcup\{P : P$ is a prime ideal of L with $U \cap P = \emptyset\} \supseteq L \setminus \downarrow\pi(U)$ by Lemma 3.1. Since $\downarrow\pi(U)$ is σ-closed, we have $L \setminus \downarrow\pi(U) \subseteq \sigma$-interior $U = \Uparrow U$. By the definition of $\pi(U)$, we obtain $\downarrow\pi(U) = \downarrow(\psi L \setminus \Uparrow U) \subseteq L \setminus \Uparrow U$, since $\Uparrow U$ is an upper set. Hence:

3.2. <u>LEMMA.</u> For any filter U in a continuous frame, $\Uparrow U = L \setminus \downarrow\pi(U)$. #

Now let $Q \in Q(\psi L)$; then for any prime ideal P of L we have $P \subseteq \downarrow Q$ iff $\sup P \in Q$; let $\mathcal{P}$ be the set of all of these P. Then $L \setminus \bigcup\mathcal{P}$ is the filter $\alpha(Q)$ generated by $L \setminus \downarrow Q$. For a pseudo-prime p we have $p \in \Uparrow \alpha(Q)$ iff for all prime ideals P the relation $p \leqslant \sup P$ implies $P \cap \alpha(Q) \neq \emptyset$ (cf. Lemma 3.1) iff $P \cap \alpha(Q) = \emptyset$, i.e., $P \in \mathcal{P}$, implies $p \nleqslant \sup P$ iff $p \notin \downarrow Q$. Thus $Q = \pi\alpha(Q)$. According to Lemma 3.2, we have $\Uparrow\alpha(Q) = L \setminus \downarrow\pi\alpha(Q) = L \setminus \downarrow Q$. Thus $\alpha(Q)$ is generated by $\Uparrow\alpha(Q)$. If U is an arbitrary filter, then $\alpha\pi(U)$ is the filter generated by $L \setminus \downarrow\pi(U)$, i.e., by $\Uparrow U$ according to Lemma 3.2.

In order to formulate what we have just shown, we record R.-E.Hoffmann's definition:

3.3. DEFINITION (R.-E.Hoffmann). For any complete lattice L we denote with $\mathrm{Filt}_\sigma L$ the poset of all filters U which are generated by their Scott interior.#

If L is a continuous lattice, then $U \in \mathrm{Filt}\, L$ is in $\mathrm{Filt}_\sigma L$ iff U is generated by $\Uparrow U$.

Our discussion showed the following result:

3.4. THEOREM. For any continuous frame L the function $\alpha : Q(\Psi L) \longrightarrow \mathrm{Filt}_\sigma(L)$ is an isomorphism betweeen continuous frames with inverse π, where $\alpha(Q)=$ filter generated by $L\setminus\downarrow Q$ and $\pi(U) = \Psi L \setminus U$. In particular, $\mathrm{Filt}_\sigma(L)$ is a frame.#

Since $Q(\Psi L)$ is the Lawson dual of $O(\Psi L)$, it follows that $\mathrm{OFilt}\ O(\Psi L) \cong \mathrm{Filt}_\sigma L$, and, as a consequence of Lawson duality, $O(\Psi L) = \mathrm{OFilt\ Filt}_\sigma L$.

At this point, my once private communication to R.-E.Hoffmann ended. His proof of the isomorphism $O(\Psi L) \cong \mathrm{OFilt\ Filt}_\sigma L$, which is his result, is different: $[H_2]$, 410, p.112.

I want to add a few comments in this memo . Firstly, the content of Section 1 above is nothing new in sofar as it is just a self-contained discussion in a special case of information contained in the Compendium, [C], pp. 312-313, 1.16-1.19. Thus in lieu of the material in Section 1 (with the possible exception of Lemma 1.2) a reference to [C] would suffice. But since spectral theory is so central, maybe a direct presentation in the spirit of the classical spectral theory with ω may be in order.

Secondly, Hoffmann's paper contains important information which is not contained in the memo above. I will comment on some of it below:

4. $\mathrm{Filt}_\sigma L$ as the injective hull of the Lawson dual OFilt L of L.

By the definition of Filt L in 3.3 it is evident that OFilt L is a subsemilattice of the continuous frame Filt_σ L.

4.1. LEMMA. If L is a continuous frame and $U \in \mathrm{Filt}\, L$, then $U = \bigvee\{V \in \mathrm{OFilt}\, L : \inf V \in U\}$.

Proof. If $v \in \Uparrow U$, then there is a $u \in U$ with $u \ll v$. Since OFilt L is a basis of σ (cf. [C], p.107, 1.14), there is a $V \in \mathrm{OFilt}\, L$ with $v \in V \subseteq \Uparrow u$. But then $u \leq \inf V$. Thus $\Uparrow U \subseteq \bigcup\{V \in \mathrm{OFilt}\, L : \inf V \leq U\}$. Since $\Uparrow U$ generates U by definition of Filt_σ L, the assertion follows.#

4.2. LEMMA. ($[H_1]$, 2.2) In Filt_σ L we have $U \ll V$ iff $\inf U \in V$.

Proof. Since directed sups in Filt L are unions, $\inf U \in V$ implies $U \ll V$. If $U \ll V$, then by Lemma 4.1 and the definition of $\ll$, there are open filters $F_1,\dots,F_n \in \mathrm{OFilt}\, L$ with $\inf F_k \leq V$ for $k = 1,\dots,n$ and $U \subseteq F_1 \vee \dots \vee F_n$.

But then $\inf U \geq \inf(F_1 \vee \dots \vee F_n) = \inf F_1 \wedge \dots \wedge \inf F_n \in V$, since V is a filter. Hence $\inf U \in V$.#

4.3. LEMMA. If $U,V,W \in \mathrm{Filt}_\sigma$ L and $U \ll V$, $W \nsubseteq V$, then there is an $F \in \mathrm{OFilt}\, L$ with $U \ll F$ and $W \nsubseteq F$.

Proof. Since $U \ll V$, by Lemma 4.2 we have $\inf U \leq V$, and since $W \nsubseteq V$ there is a $w \in W \setminus V$. Now $L \setminus \uparrow w$ is a Scott open neighborhood of $\inf U$, and since OFilt L is a basis for σ (loc.cit.) there is an $F \in$ OFilt L with $\inf U \in F$ and $w \notin F$. This F is the desired one.#

4.4. LEMMA. Let H be a continuous lattice and S a subsemilattice containing 1 and satisfying the following condition

(*) Whenever $⇑x \setminus \uparrow y \neq \emptyset$, then $S \cap (⇑x \setminus \uparrow y) \neq \emptyset$.

Then S is λ-dense in L.

Proof. We claim

(**) Whenever $F \setminus \uparrow y \neq \emptyset$ for some $F \in$ OFilt L, then $S \cap (F \setminus \uparrow y) \neq \emptyset$.

Indeed, let $f \in F \setminus \uparrow y$. Since L is continuous, there is an $x \in F$ with $x \ll y$ (see [C], p.104, 1.10). By (*) we find an $s \in S$ with $x \ll s$, $y \nleq s$. Sinc then $s \in F$, condition (**) is proved. Now we claim

(***) Whenever $F \setminus (\uparrow y_1 \ldots \uparrow y_n) \neq \emptyset$ for $F \in$ OFilt L and $y_k \in L$, then
$S \cap (F \setminus (\uparrow y_1 \cap \ldots \cap \uparrow y_n) \neq \emptyset$.

Proof: By (**) we find elements $s_k \in S \cap (F \setminus \uparrow y_k)$ for $k = 1, \ldots, n$. But then $s = s_1 \ldots s_n$ is in $S \cap F$ since S is a semilattice and F a filter, and $s \notin \uparrow y_1 \cap \ldots \cap \uparrow y_n$. This shows (***).

Finally, the sets $F \setminus (\uparrow y_1 \cap \ldots \cap \uparrow y_n)$, $F \in$ OFilt L, $y_1, \ldots, y_n \in L$ together with the sets $⇑x$, $x \in L$ form a basis for λ. Then by (***) and since $1 \in S$ we know that S is λ-dense in L.#

4.5. PROPOSITION. For a continuous frame L, the semilattice OFilt L is sup-dense and λ-dense in Filt_σ L.

Proof. Lemma 4.1 shows that OFilt L is sup-dense in Filt L. Lemma 4.3 shows that condition (*) of Lemma 4.4 is satisfied for H = Filt L and S = OFilt L. Then by Lemma 4.4, OFilt L is λ-dense in Filt_σ L.#

Now we turn to J.D.Lawson, Obtaining the T_0-essential hull, in "Continuous Lattices and their Representations" , Proceedings of the Workshop on Continuous Lattices July 1982 in Bremen, Marcel Dekker 1983/84 and find as Corollary 8 the following result

4.6. PROPOSITION. If H is a continuous lattice and X a subset which is sup-dense and not contained in a proper λ-closed subsemilattice with identity inside H, then (H, σ) is the essential (hence injective) hull of $(X, \sigma | X)$.#

As a consequence of Propositions 4.5 and 4.6 we obtain

4.7. THEOREM (R.-E.Hoffmann, [H_1], Theorem 2.5). Filt_σ L is the injective hull of OFilt L (relative to the Scott topology on Filt_σ L and the induced topology on OFilt L).

R.-E.H.'s proof of 4.5 is even shorter, combining 4.3 and 4.4. But 4.4 may be of independent use as a sufficient (and necessary) condition for λ-density of a subsemilattice in a continuous lattice.

If we follows H_1 in abbreviating OFilt by D and injective hull by I, then the formula $O(\Psi L) = D(\mathrm{Filt}\ L)$ after 3.4 above by 4.7 yields Hoffmann's formula $O\Psi L = DIDL$.

5. The gamma-spectrum of a continuous lattice.

The gamma spectrum of a continuous frame L is the set $\mathrm{Gam}(L) = \{x \in L : x = \sup(\downarrow x \cap \Psi L)\}$. Its elements are also called gamma-elements (cf. R.-E.Hoffmann, Math.Z. 179 ,loc.cit., and SCS-memo 1-9-83). The gamma-spectrum is a complete lattice and the MacNeille completion of $\Psi(L)$ since is inf- and sup-dense in Gam(L). R.-E.H. defines a topology for the gamma-spectrum, the so called gamma topology. In SCS-memo 1-9-83 he proves that it induces on the pseudospectrum $\Psi(L)$ the topology of $\Psi(L)$, viz. $\omega|\Psi(L)$. The gamma-spectrum is the essential hull of Spec (L) (and then also of $\Psi(L)$). The pseudo-spectrum is one topic, the gamma-spectrum another, presumably bigger one (since the gamma-spectrum is bigger than the pseudo-spectrum). But the topic of this memo was that, if one wants to talk about the pseudo-spectrum and its interesting properties, one can do so without knowing much about the gamma spectrum. I do know the gamma-spectrum reasonably well, since I refereed R.-E.H.MZ.179 and recommended publication; I do recommend reading it for the internal interest of the gamma-spectrum. That I do not recommend basing the theory of the pseudo-spectrum on the gamma-spectrum is a separate story.