SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC: The lower topology for continuous lattices is a monadic functor


REFERENCES: [1] Alan Day, Filter monads, continuous lattices and closure systems. Canad. J. Math. 27 (1975), 50 - 59.

[2] G. Gierz et al., A Compendium of Continuous Lattices, Springer-Verlag, 1980.

[3] Oswald Wyler, Algebraic theories of continuous lattices. Continuous Lattices, LNM 871 (1981), 390 - 413.

[4] Oswald Wyler, Compact ordered spaces and ~~the~~ prime Wallman compactifications. Preprint, Carnegie-Mellon Univ., 1983.


Alan Day [1] obtained the category CL of continuous lattices as the category of algebras for the open filter monad on TOP ; I showed in [3] that the forgetful functor $S : CL \longrightarrow TOP$ for this representation is provided by Scott topologies. Thus Scott topologies provide a monadic functor from CL to TOP .

In the present note, I obtain an analogous result for the lower topologies of continuous lattices, using the closed filter monad on TOP . This is based on the main result of [4], characterizing compact ordered spaces (compact pospaces in the Compendium) as algebras for the prime closed filter monad.

Notations of [3] and of the Compendium will be used as much as possible. In particular, $P_p : ENS^{op} \longrightarrow LAT$ and $P_m : ENS^{op} \longrightarrow MSL$ assign to a set S the lattice and the meet semilattice (with intersections) of subsets of S . These functors have adjoints $G_p : LAT^{op} \longrightarrow ENS$ and $G_m : MSL^{op} \longrightarrow ENS$ on the right, obtained from sets of prime filters and sets of filters. All four functors operate on morphisms by inverse images; they are subfunctors of lifted powerset functors, and the adjunctions are exponential, i.e. f corresponds to g by one of the adjunctions iff always $a \in f(x) \Longleftrightarrow x \in g(a)$ . These adjunctions define the ultrafilter monad on sets, with compact Hausdorff spaces as algebras, and the filter monad with continuous lattices as its algebras.

Lifting these functors to TOP , we have functors $\Gamma : TOP^{op} \longrightarrow LAT$ and $\Gamma_m : TOP^{op} \longrightarrow MSL$ which assign to a topological space X the lattice $\Gamma X$ and the meet semilattice $\Gamma_m X$ of closed sets, with intersections as meets. If we provide $G_p L$ and $G_m L$ with the hull-kernel topology, with sets $a^{\#} = \{\varphi : a \in \varphi\}$ as a subbasis of closed sets, we obtain lifted functors which we denote by $\Sigma : LAT^{op} \longrightarrow TOP$ and $\Sigma_m : MSL^{op} \longrightarrow TOP$ . The "greek letter" functors are again subfunctors of lifted powerset functors; they operate on morphisms by inverse images.

PROPOSITION 1. <u>$\Sigma$ and $\Gamma$, and $\Sigma_m$ and $\Gamma_m$, are adjoint on the right, with exponential adjunctions.</u>

<u>Proof</u>. If $f : X \longrightarrow G_m L$ corresponds to $g : L \longrightarrow P_m X$ , then $g(a) = f^{-1}(a^{\#})$ for $a \in L$ . Thus $f : X \longrightarrow \Sigma_m L$ is continuous iff $g$ maps $L$ into $\Gamma_m X$ , and we have a natural bijection between maps $f : X \longrightarrow \Sigma_m L$ in TOP and maps $g : L \longrightarrow \Gamma_m X$ in MSL .

The proof for $\Sigma$ and $\Gamma$ is exactly analogous.

We now have four monads from adjunctions on the right, the filter monad $\underline{F}$ and the ultrafilter monad $\underline{U}$ on sets, and the <u>closed filter monad</u> $\underline{G}$ and the <u>prime closed filter monad</u> $\underline{W}$ on topological spaces. These monads are related.

If $R : TOP \longrightarrow ENS$ and $J : LAT \longrightarrow MSL$ are the forgetful functors, then

$$J P_p = P_m , \quad J \Gamma = \Gamma_m , \quad R\Sigma = G_p , \quad R\Sigma_m = G_m .$$

We also have natural embeddings

$$\lambda : \Gamma \longrightarrow P_p R^{op} \quad \text{and} \quad \rho : \Sigma \longrightarrow \Sigma_m J^{op} ,$$

adjoint on the right to $id(G_p)$ and $id(\Gamma_m)$ . One verifies easily that these embeddings produce four morphisms of monads (see e.g. [3; 1.5])

$$\begin{array}{ccc} \underline{G} & \xleftarrow{(R,\ G_m J^{op} \lambda^{op})} & \underline{F} \\ \uparrow {\scriptstyle (Id,\ \rho \Gamma^{op})} & & \uparrow {\scriptstyle (Id,\ R\rho P_p^{op})} \\ \underline{W} & \xleftarrow{(R,\ G_p \lambda^{op})} & \underline{U} \end{array}$$

which form a commutative square. The mappings $G_p \lambda_X$ and $G_m J \lambda_X$ assign to an ultrafilter or filter $\varphi$ on $X$ the prime closed filter or closed filter of all closed sets in $\varphi$ ; these mappings are surjective.

$\underline{W}$-algebras $(X, \alpha)$ are compact ordered spaces $(Z, \leq)$ . The topology of $X$ is the upper topology; we have $x \leq y$ iff $x \in cl_X \{y\}$ . The $\underline{U}$-algebra structure $\alpha \cdot G_p \lambda_X$ is convergence of ultrafilters in the compact Hausdorff space $Z$ , and $Z$ has the patch topology for $X$ . Morphisms of $\underline{W}$-algebras are continuous maps of the compact Hausdorff spaces which preserve order. We refer to [4] for proofs of these and related results.

The spaces $\Sigma L$ and $\Sigma_m L$ are $T_0$ spaces, in fact compact ordered spaces with the upper topology. In both spaces, order is dual to the set inclusion for filters in $L$. The closure of $\{\varphi\}$ in either space is the intersection of all $a^\#$ with $a \in \varphi$, consisting of all prime filters or filters in $L$ which contain $\varphi$.

After these preliminaries, we turn to our main result.

PROPOSITION 2. <u>G-algebras are continuous lattices provided with the lower topology; morphisms of G-algebras are morphisms of the underlying continuous lattices.</u>

<u>Proof</u>. Using the morphisms of monads displayed above, a $\underline{G}$-algebra $(X,\alpha)$ is a compact ordered space $(Z,\leq)$ for which $X$ has the upper topology, and a continuous lattice $L = (RX, \alpha \cdot G_m \lambda_X)$. By commutativity of the square displayed above, both $(Z,\leq)$ and $L$ have the same underlying compact space, i.e. the topology of $Z$ is the Lawson topology of $L$.

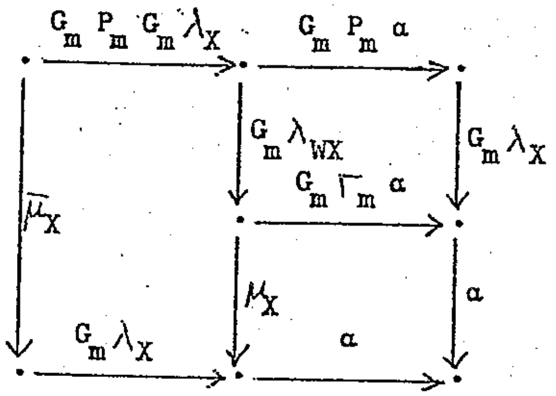
For a filter $\varphi$ on $L$, we have $\alpha(\mathrm{cl}_X \varphi) = \inf_L \mathrm{adh}_Z \varphi$. In particular, $x \cap y = \alpha(\uparrow \mathrm{cl}_X \{x,y\})$. If $x \leq y$ in $X$, then $\mathrm{cl}_X \{x,y\} = \mathrm{cl}_X \{y\}$; thus $x \cap y = \inf_L \mathrm{adh}_Z \uparrow \{y\} = y$, and $y \leq x$ in $L$. It follows that a decreasing set for $L$ is increasing for $X$, and hence that the topology of $X$, the upper topology for $Z$, is finer than the lower topology for $L$.

If $y \leq x$ in $L$, then the sets $\uparrow x$ and $\uparrow y$ are closed for $X$, and $\uparrow x \subset \uparrow y$. Thus $\mathrm{cl}_X \uparrow\uparrow x \leq \mathrm{cl}_X \uparrow\uparrow y$ in $\Sigma_m \Gamma_m X$. The structure $\alpha$ of $(X,\alpha)$ is a morphism of $\underline{G}$-algebras, hence of the underlying compact ordered spaces; thus it preserves order. Now $\inf_L \mathrm{adh}_Z \uparrow\uparrow x = x$, and similarly for $y$; thus $x \leq y$ in $X$. We conclude that $(Z,\leq)$ has the dual order of $L$; thus its upper topology, the topology of $X$, is the lower topology of $L$.

Conversely, let $L$ be a continuous lattice with $\underline{F}$-algebra structure $\beta$, with $\beta(\varphi) = \inf \mathrm{adh}\, \varphi = \sup (\inf A)_{A \in \varphi}$ for a filter $\varphi$ on $L$; this depends only on the Lawson-closed sets in $\varphi$. If $X$ is $L$ with the lower topology, then $\mathrm{cl}_X \varphi$ consists of the sets $\uparrow A$ with $A$ Lawson-closed in $\varphi$. Passing from $A$ to $\uparrow A$ does not change $\inf A$; thus $\beta(\varphi) = \beta(\mathrm{cl}_X \varphi)$. It follows that $\beta = \alpha \cdot G_m \lambda_X$ for a mapping $\alpha : G_m \Gamma_m X \longrightarrow X$.

$\beta$ and $G_m \lambda_X$ are continuous for the compact Hausdorff topologies, i.e. the Lawson topologies, of the spaces involved, and $G_m \lambda_X$ is surjective. It follows that $\alpha$ is continuous for the Lawson topologies. If $\varphi \leq \psi$ in $\Sigma_m \Gamma_m X$, then $\psi$ is coarser than $\varphi$, so that $\beta(\psi) \leq \beta(\varphi)$ in $L$, hence $\alpha(\varphi) \leq \alpha(\psi)$ in $X$. Thus $\alpha : \Sigma_m \Gamma_m X \longrightarrow X$ is continuous.

Clearly $\alpha(\uparrow cl_X \{x\}) = \beta(\uparrow\{x\}) = x$ for $x \in X$; thus $\alpha \bar{\eta}_X = id_X$ for the unit $\bar{\eta}$ of $\underline{G}$. If $\mu$ and $\bar{\mu}$ are the multiplications of $\underline{F}$ and $\underline{G}$, and we put $W = \Sigma_m \Gamma_m^{op}$, then we have the following diagram of mappings.




The outer square commutes because $\alpha \cdot G_m \lambda_X$ is an $\underline{F}$-algebra structure, and the rectangle at left because $(R, G_m J^{op} \lambda^{op})$ is a morphism of monads. The upper small rectangle commutes by naturality of $\lambda$. Since $G_m \lambda_X$, and hence also $G_m P_m G_m \lambda_X$, is surjective, the lower rectangle also commutes; thus $\alpha$ is the desired $\underline{G}$-algebra structure of $X$.

The morphism $(R, G_m J^{op} \lambda^{op}) : \underline{F} \longrightarrow \underline{G}$ of monad maps a morphism $f$ of $\underline{G}$-algebras into a morphism $R f$ of the underlying continuous lattices. Conversely, if $f : L \longrightarrow M$ is a morphism of continuous lattices, then $f$ is continuous for the Lawson topologies and preserves order; thus $f : X \longrightarrow Y$ is continuous for $L$ and $M$ with the lower topologies. If $\alpha$ and $\beta$ are the $\underline{G}$-algebra structures of $X$ and $Y$, then

$$f \cdot \alpha \cdot \Sigma_m \lambda_X = \beta \cdot \Sigma_m \lambda_Y \cdot \Sigma_m P_m f = \beta \cdot \Sigma_m \Gamma_m f \cdot \Sigma_m \lambda_X ,$$

since $f$ is a morphism of $\underline{F}$-algebras and by naturality of $\lambda$. Since $\Sigma_m \lambda_X$ is surjective, $f : (X,\alpha) \longrightarrow (Y,\beta)$ follows. This completes the proof.

Monadicity of the lower topology functor has its usual consequences: the functor preserves categorical limits including products, and we have a free $\underline{G}$-algebra, or a free continuous lattice $K_m \Gamma_m X$, for every topological space, so that if $L$ is a continuous lattice, then every map $f : X \longrightarrow L$, which is continuous for the lower topology of $L$, extends to a unique morphism $g : K_m \Gamma_m X \longrightarrow L$ of continuous lattices, with $f = g \bar{\eta}_X$.