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TOPIC: Continuous Lattices, General Convexity spaces and a fixed point theorem

General convexity and continuous lattice theory meet in what here is called continuous convexities ("compact convexities" might be a better denomination). By this we mean a family $\mathcal{K}$ of non-empty closed subsets of a compact Hausdorff space X which is a closure system (i.e. $K_i \in \mathcal{K}$ and $\bigcap K_i \neq \emptyset$ imply $\bigcap K_i \in \mathcal{K}$) and regular (i.e. every $K \in \mathcal{K}$ has a neighborhood base belonging to $\mathcal{K}$). Such a collection $\mathcal{K}$ considered as a set ordered by inclusion is in fact a (dually) complete continuous semilattice L in the terminology of [Comp], and every complete continuous semilattice has a canonical representation as a continuous convexity.

In the first part of these notes we discuss these connections. We owe a lot to the dissertations of Jamison [J] and Tiller [Ti 1,2]. Then we present a version of Ky Fan's fixed point theorem [KF] due to Wissel [Wi] in the language of continuous convexities. It turns out that this theorem of Wissel covers the corresponding theorem 5.2 due to Wieczorek [Wk]. It is an open problem whether the two theorems are equivalent.

Please, communicate to me any observations on this Note, which is to be considered an informal one.

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Remark: No references are given in the text. This list serves as a global reference.

1. Complete continuous semilattices

1.1. COMPLETE $\vee$-SEMILATTICES A complete $\vee$-semilattice is a non-empty ordered set such that
- every non-empty subset has a join, and
- every filtered subset has a meet.

Here a subset F of an ordered set is called filtered if $F \neq \emptyset$ and if every finite subset of F has a lower bound in F.

1.2 WAY-ABOVE In a complete $\vee$-semilattice L we may introduce the way-above relation $\gg$ in the following way:

$b \gg a$ if for every filtered subset F of L with $a \geq \inf F$ there is a $c \in F$ such that $b \geq c$

1.3. COMPLETE-CONTINUOUS SEMILATTICES A complete-continuous semilattice is a complete $\vee$-semilattice such that for all $a \in L$:

$$a = \inf \{ b \,;\, b \gg a \}$$

REMARKS. A complete (=continuous) $\vee$-semilattice L always has a top element, but need not have a bottom element. By adjoining a new element 0 as smallest element, one obtains a complete lattice L_0.

It is only for convenience, that we use the order opposite to that in the Compendium. So, L is a complete-continuous $\vee$-semilattice iff L^{op} is a complete-continuous $\wedge$-semilattice in the sense of the Compendium, i.e. iff $(L_0)^{op}$ is a continuous lattice in the sense of the Compendium. In particular, we consider two intrinsic topologies:

1.4 SCOTT TOPOLOGY σ_L (cf. Compendium II-1) On a complete-continuous $\vee$-semilattice L, the open sets for the Scott topology σ_L are the lower ends $\mathcal{U}$ of L such that for every $a \in \mathcal{U}$ there is a $b \in \mathcal{U}$ with $b \gg a$. A basis for the Scott topology is given by the sets of the form

$$\Downarrow b = \{ a \in L \,;\, b \gg a \} \,, \quad b \in L$$

A function $f : L \to M$ of complete-continuous $\vee$-semilattices is continuous for the respective Scott topologies iff f is monotone and preserves meets of filtered sets.

1.5 LAWSON TOPOLOGY λ_L (see Compendium III-1) On a complete-continuous $\vee$-semilattice L, the Lawson topology λ_L is the topology generated by σ_L and ω_L, where ω_L is the topology whose closed sets are generated by the principal ideals $\Downarrow a = \{ b \in L \,;\, b \geq a \}$, $a \in L$

The Lawson topology is compact, Hausdorff and makes the operation join $(a,b) \mapsto a \vee b : L \times L \to L$ continuous. A $\vee$-homomorphism $f : L \to M$ of complete-continuous $\vee$-semilattices is continuous for the respective Lawson topologies iff it preserves joins of non-empty sets and meets of filtered sets.

1.6 EXAMPLE: HYPERSPACES Let X be a compact Hausdorff space and $\mathcal{C} = \mathcal{C}(X)$ the collection of all closed non-empty subsets of X ordered by inclusion. Then, $\mathcal{C}$ is a complete-continuous $\vee$-semilattice; indeed, for $A, B \in \mathcal{C}$ we have

$$B \gg A \quad \text{iff} \quad B \text{ is a neighborhood of } A.$$

2. Continuous Convexities

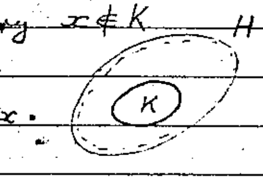
Complete-continuous $\vee$-semilattices arise as certain systems of closed subsets of compact Hausdorff spaces. This has been discovered implicitely by R.J. Jamison and has been discussed extensive= ly by J. Tiller.

Throughout this section let X be a compact Hausdorff space, and $\mathcal{C}$ the collection of all non-empty closed subsets.

2.1 DEFINITION. A collection $\mathcal{K} \subseteq \mathcal{C}$ of non-empty closed sub= sets of X is called a continuous convexity, and $(X, \mathcal{K})$ is called a continuous convexity space, if the following ~~two~~ conditions are satisfied:

(C) Closure property: For every family $(K_i)_i$ in $\mathcal{K}$ with non-empty intersection, one has $\bigcap_i K_i \in \mathcal{K}$.

(R) Regularity: For every $K \in \mathcal{K}$ and every $x \notin K$ there is a neighborhood H of K with $H \in \mathcal{K}$ and $x \notin H$.



For continuous convexity spaces the following separation axioms are of interest:

(T_0) For any two distinct elements in X there is a set $K \in \mathcal{K}$ containing just one of them

(T_1) $\{x\} \in \mathcal{K}$ for every $x \in X$.

2.2 CLOSURE SYSTEMS The property (C) says that $\mathcal{K}$ is a closure system on $\mathcal{C}$ according to the following definition:
A subset M of a complete $\vee$-semilattice L is called a closure system if $\bigwedge_i h_i \in M$ for every family $(h_i)_i$ in M such that $\bigwedge h_i$ exists in L.
Such a closure system $M \subseteq L$ determines a closure operator

$$c : L \to L \quad \text{by} \quad c(a) = \bigwedge \{h \in H ; a \leq h\}$$

Cleary, $c(L) = M$ and c satisfies the following properties:

$$a \leq c(a), \quad c(c(a)) = c(a), \quad a \leq b \Rightarrow c(a) \leq c(b)$$

As a subset of L, M is partially ordered and every non-empty family $(h_i)_i$ in M has a join in M (which may be bigger than the join in L), namely

$$\bigvee_i^M h_i = c(\bigvee_i^L h_i)$$

More generally, for every non-empty family $(s_i)_i$ in L, one has

$$\bigvee_i^M c(s_i) = c(\bigvee_i^L s_i)$$

which means that the map $c : L \to M$ preserves joins of non-empty families.
As every filtered subset of M has a meet in L and as this meet also belongs to M by definition, M also has filtered meets and, hence, is a complete $\vee$-semilattice. We now quote from the Compendium:

2.3 PROPOSITION. For a closure system M in a complete-continuous $\vee$-semilattice L, the following properties are equivalent:

(i) The closure operator $c: L \to M$ preserves meets of filtered sets.

(ii) $b \gg a$ implies $c(b) \gg c(a)$.

(iii) $a = \bigwedge \{b \in M;\ b \gg a\}$ for all $a \in M$.

Moreover, if these conditions are satisfied, M is a complete-continuous $\vee$-semilattice and the closure operator $c: L \to M$ is continuous for the respective Lawson topologies; for $a, b \in M$ one has $b \gg a$ in L iff $b \gg a$ in M. The Scott topology on M is the restriction of the Scott topology on L.

2.4 CONVEXITIES AS SEMILATTICES. We apply 2.3 to the case where L is the collection $\mathcal{C}$ of all non-empty closed subsets of a compact Hausdorff space X, and $M = \mathcal{K}$ a continuous convexity on X. We already noticed that $\mathcal{K}$ is a closure system. We denote with $c_{\mathcal{K}}$ the associated $\mathcal{K}$-closure operator:

$$c_{\mathcal{K}}(A) = \bigcap \{K \in \mathcal{K};\ A \subseteq K\} \quad \text{for } \emptyset \neq A \subseteq X.$$

$\mathcal{K}$ is a complete $\vee$-semilattice: Meets in $\mathcal{K}$ are intersections (as far as they are non-empty), and joins are formed according to

$$\bigvee_i^{\mathcal{K}} K_i = c_{\mathcal{K}}\left(\bigcup_i K_i\right)$$

We now observe that the regularity (R) of $\mathcal{K}$ just means that condition (ii) of proposition 2.4 is satisfied, and we conclude

PROPOSITION. Let $\mathcal{K}$ be a continuous convexity on X. Then $\mathcal{K}$ is a complete continuous $\vee$-semilattice. The $\mathcal{K}$-closure operator $c_{\mathcal{K}}: \mathcal{C} \to \mathcal{K}$ is continuous for the Lawson topologies on $\mathcal{C}$ and $\mathcal{K}$, respectively. For $K, H \in \mathcal{K}$, we have $H \gg K$ iff H is a neighborhood of K. The Scott topology on $\mathcal{K}$ is the subspace topology induced by the Scott topology on $\mathcal{C}$.

If we restrict the $\mathcal{K}$-closure operator to singletons, we obtain a map

$$x \mapsto c_{\mathcal{K}}(x) : X \to \mathcal{K}.$$

This map is composed by the map $x \mapsto \{x\} : X \to \mathcal{C}$ and $c_{\mathcal{K}}: \mathcal{C} \to \mathcal{K}$. Both of these maps are continuous with respect to the Lawson topologies on $\mathcal{C}$ and $\mathcal{K}$. Thus it is continuous, too. The injectivity of this map is equivalent to the separation axiom (T_0). We have:

2.5 PROPOSITION. For a continuous convexity $\mathcal{K}$ on a compact Hausdorff space X, the natural map

$$x \mapsto c_{\mathcal{K}}(x) : X \to \mathcal{K}$$

is continuous, where $\mathcal{K}$ is endowed with the Lawson topology. It is an embedding iff $\mathcal{K}$ is a T_0-convexity.

Before listing examples of convexities, let me indicate that every complete-continuous $\vee$-semilattice may be viewed as a convexity:

2.6 COMPLETE-CONTINUOUS SEMILATTICES AS CONVEXITIES. We first notice: Let $\mathcal{K}$ be a continuous convexity on X. For every $K \in \mathcal{K}$ we have

$$K = \bigcup \{c_{\mathcal{K}}(x);\ x \in X\},$$

i.e. the set of all $c_{\mathcal{K}}(x)$, $x \in X$, is a λ-closed join-dense subset of $\mathcal{K}$. It follows that every irreducible element of $\mathcal{K}$ is of the form $c_{\mathcal{K}}(x)$ for some $x \in X$. This yields the idea of a minimal representation of a complete-continuous $\vee$-semilattice as a convexity:

Let L be a complete-continuous $\vee$-semilattice. Denote by X the closure of the set of all $\vee$-irreducibles in L with respect to the Lawson topology. With respect to the topology induced by the Lawson topology on L, X is a compact Hausdorff space. To every element $a \in L$ we associate the closed subset

$$h(a) = \{x \in X;\ x \leq a\} \subseteq X,$$

and we denote by $\mathcal{K}$ the collection of all $h(a)$, $a \in L$. As $b \gg a$ implies that $h(b)$ is a neighboorhood of $h(a)$, $\mathcal{K}$ is a continuous T_0-convexity on X, and $\mathcal{K}$ is isomorphic to L as an ordered set via the map h.

2.7 EXAMPLES. a) $\boxed{\text{COMPACT CONVEX SETS}}$. The collection of all non-empty closed convex subsets of a compact convex set in a locally convex topological vector space is a continuous T_1-convexity, in fact the prime example.

b) $\boxed{\text{CLOSED CONNECTED SETS.}}$ Let X be a compact connected space and $\mathrm{Conn}(X)$ the set of all non empty closed connected subsets. The intersection of a filtered family of connected sets is connected. If in addition the intersection of any two closed connected sets is connected (in this case, X is called hereditarily unicoherent), then $\mathrm{Conn}(X)$ satisfies the closure property (C). It is rather straightforward that $\mathrm{Conn}(X)$ is regular iff X is locally connected. Thus, $\mathrm{Conn}(X)$ is a continuous convexity iff X is a compact, locally connected, hereditarily unicoherent space (such spaces are called trees).

a) $\boxed{\text{CLOSED SUBSEMILATTICES}}$ Let L be a complete-continuous $\vee$-semilattice. The set $\mathcal{S}$ of all non-empty closed $\vee$-subsemilattices (closed with respect to the Lawson topology) is a continuous T_1-convexity. The same holds for all order convex closed subsemilattices. (Indication of the proof of the regularity of $\mathcal{S}$: Let $S \in \mathcal{S}$ and $a \notin S$. Let $b = \max(S \cap \downarrow a)$. Choose $c \gg b$ such that $c \not\leq a$ and $d \gg a$ such that $d \notin S$. Then $T := \downarrow c \vee (L \setminus \uparrow d)$ is a Lawson closed neighborhood of S belonging to $\mathcal{S}$ with $a \notin T$.)

d) $\boxed{\text{CLOSED RELATIONS}}$ Let X and Y be compact Hausdorff spaces. Like the set of all non-empty closed subsets of a compact Hausdorff-space, the set $\mathcal{R}(X,Y)$ of all closed non-empty relations $R \subseteq X \times Y$ is a continuous convexity.

In the case of $X = Y$, the collections of closed symmetric and reflexive relations are continuous convexities on $X \times X$. This does not hold for the collection of all closed transitive relations on X. For example, every transitive relation on the unit interval $I = [0,1]$ which is a neighborhood of the diagonal $\Delta \subseteq I \times I$ is all of $I \times I$; thus, the regularity axiom (R) is not satisfied. But we have the following:

The equivalence relations E on a compact Hausdorff space X which are neighboorhoods of the diagonal $\Delta \subseteq X \times X$ correspond exactly the the partitions of X into open subsets. Because of the compactness of X, there can only be finitely many equivalence classes, and E is closed (and open). Thus, E is a neighborhood of Δ iff the quotient space X/E is finite. From this we infer:

If $\mathcal{E}_0$ is a collection of closed-and-open equivalence relations on X, i.e. X/E is finite for each $E \in \mathcal{E}_0$, and if $\mathcal{E}$ is the collection of all intersections of members in $\mathcal{E}_0$, then $\mathcal{E}$ is a continuous convexity on $X \times X$; each continuous convexity $\subseteq \mathcal{E}$ of closed equivalence relations can be obtained in this way. In particular, X/E is totally disconnected for each $E \in \mathcal{E}$.

The collection of all closed equivalence relations on X is a continuous convexity on X iff X/E is totally disconnected for each closed equivalence relation. Of course, this implies that X is totally disconnected. But total disconnectedness is not sufficient for this to be true. E.g. the Cantor set does not have this property.

e) **CLOSED CONGRUENCES** Let us consider a compact universal algebra A, i.e. a compact Hausdorff space together with a family of continuous finitary operations. Applying our results on closed equivalence relations to congruences, we obtain:

<u>The collection $\mathcal{F}$ of all closed congruence relations on a compact universal algebra A is a continuous convexity iff A/ϱ is profinite for each $\varrho \in \mathcal{F}$.</u>

Recall that a compact algebra A is profinite if the continuous homomorphisms onto finite algebras separate the points of A. In the case of compact lattices (?), compact semilattices, compact semigroups, and compact groups, profiniteness is already granted by the total disconnectedness of the space. In the case of a compact groups, but not in the other cases, every quotient of a profinite group is profinite.

f) **COMPACT ORDERED SPACES** Let X be a compact ordered space, i.e. a compact space endowed with a closed order. The collection of all non-empty lower (upper) ends of X is a continuous convexity which is T_0 but not T_1 except for the case of a trivial order on X. In fact, its exactly the distributive complete-continuous $\vee$-semilattices in which the set X of $\vee$-irreducibles is Lawson-closed that can be represented by a convexity of this type.

The non-empty Lawson-closed upper ends of a complete continuous $\vee$-semilattice L are a continuous convexity of the above type living on L endowed with its Lawson topology. As the Lawson-closed upper ends are exactly the Scott-closed sets, we see that the Scott-closed non-empty subsets of L form a continuous convexity. The same holds for every compact $\vee$-semilattice.

2.8 **INDUCED CONVEXITIES.** Let $\mathcal{K}$ be a continuous convexity on a compact Hausdorff space X. For any closed subspace $Y \subseteq X$, let

$$\mathcal{K}|_Y = \{ K \cap Y;\ K \in \mathcal{K} \text{ such that } K \cap Y \neq \emptyset \}$$

Then $\mathcal{K}|_Y$ is a continuous convexity on Y, called the convexity induced on Y by $\mathcal{K}$. In the case where $Y \in \mathcal{K}$, then $\mathcal{K}|Y$ is just the set of all $K \in \mathcal{K}$ with $K \subseteq Y$.

2.9 **NORMALITY** A continuous convexity satisfies:

(T_4) For $K_1, K_2 \in \mathcal{K}$ with $K_1 \cap K_2 = \emptyset$ there are neighborhoods L_1, L_2 in $\mathcal{K}$ with $L_1 \cap L_2 = \emptyset$.

Indeed, by regularity, every neighborhood of $K_i \in \mathcal{K}$ contains a neighborhood $L_i \in \mathcal{K}$.

2.10 **HAUSDORFFNESS** $T_1 \Rightarrow T_2$

2.11 **REMARK** Let $\mathcal{K}$ be a T_1 continuous convexity on X. Then the Scott topology of $\mathcal{K}$ induces on X the original topology. (Proof: 2.10)

4. Upper semicontinuous functions and closed relations

Throughout, let $\mathcal{K}$ be a continuous convexity on a compact Hausdorff space X. Let T be any topological space.

A set valued map $f : T \to \mathcal{K}$ is called *upper semicontinuous*, if for every $t_0 \in T$ and every neighborhood U of $f(t_0)$ in X, the set $\{t \in T;\ f(t) \subseteq U\}$ is a neighborhood of t_0. [As $f(t_0) \in \mathcal{K}$ and as every member of $\mathcal{K}$ has a neighborhood basis in $\mathcal{K}$ by regularity, we may choose $U \in \mathcal{K}$ in this definition. From the definition of the Scott topology on $\mathcal{K}$, one immediately sees that upper semicontinuity in the above sense is equivalent to the continuity of f with respect to the Scott topology σ on $\mathcal{K}$; we have

4.1 PROPOSITION. *A multivalued map $f : T \to \mathcal{K}$ is upper semicontinuous iff and only if $f : T \to (\mathcal{K}, \sigma)$ is continuous.*

4.2 CLOSED RELATIONS. To every set valued map

$$f : T \to \mathcal{P}(X)$$

we associate its graph, i.e. the relation

$$R_f = \{(t, x);\ x \in f(t)\} \subseteq T \times X;$$

and every relation $R \subseteq T \times X$ defines set valued map $f_R : T \to \mathcal{P}(X)$ defined by

$$f_R(t) := R[t] := \{x;\ (t, x) \in R\}$$

A relation R is called a *$\mathcal{K}$-relation* if $R[t] \in \mathcal{K}$ for every $t \in T$. One easily verifies:

Under the above correspondance, the graphs of upper semicontinuous functions $f : T \to \mathcal{K}$ are exactly the closed $\mathcal{K}$-relations $R \subseteq T \times X$.

4.3 The family of closed $\mathcal{K}$-relations on $T \times X$ is regular, provided that T is a regular space. Indeed, let R be a closed $\mathcal{K}$-relation and $(t_0, x_0) \notin R$. Choose a neighborhood $K \in \mathcal{K}$ of $R[t_0]$ not containing x_0 (which is possible by the regularity of $\mathcal{K}$). By the upper semicontinuity of $t \mapsto R[t]$, there is a neighborhood V of t_0 such that $R[t] \subseteq \operatorname{int}(K)$ for all $t \in V$. By the regularity of the space T, we may choose V to be a closed neighborhood. It is clear now that

$$R' = V \times K \cup (X \setminus \operatorname{int} V) \times X$$

is a closed $\mathcal{K}$-relation not containing (t_0, x_0), and R' is a neighborhood of R. The same holds for $\mathcal{K}_0 = \mathcal{K} \cup \{\emptyset\}$ instead of $\mathcal{K}$. We conclude:

For a compact Hausdorff space T, the set of all upper semicontinuous functions $f : T \to \mathcal{K}_0$ is a continuous lattice.

5. A fixed point theorem

Our motivation is coming from convexity theory: Let X be a compact convex set embeddable in a locally convex topological vector space and $\mathcal{K}$ the collection of all non-empty closed convex subsets of X. Using Brouwer's fixed point theorem, Kakutani has proved that for every upper-semicontinuous map $f: X \to \mathcal{K}$ there is an $x \in X$ such that $x \in f(x)$ provided that X is finite dimensional. Ky Fan has extended this result to the infinite dimensional case. It is this extension (of the fixed point theorem) from the finite to the infinite-dimensional case that we want to phrase now in the abstract setting of continuous convexities.

Now, let $\mathcal{K}$ be a continuous convexity on a compact Hausdorff space X. Recall, that for a given non-empty finite subset F of X, the smallest member of $\mathcal{K}$ containing F is called a $\mathcal{K}$-polytope. All $\mathcal{K}$-polytopes P are endowed with the induced convexity $\mathcal{K}/P$.
We shall say that $\mathcal{K}$ has the fixed point property, if for every upper semicontinuous function $f: X \to \mathcal{K}$ there is a fixed point, i.e. a point x_0 such that $x_0 \in f(x_0)$. Our main result is:

5.1 THEOREM. If every $\mathcal{K}$-polytope P has the fixed point property (with respect to the induced alignment), then $(X, \mathcal{K})$ itself has the fixed point property.

For the proof we proceed by several steps.

Let X and Y be compact Hausdorff spaces and $\varrho \subseteq X \times Y$ a (closed) closed relation. For $A \subseteq X$, let

$$\varrho[A] = \bigcup_{t \in A} \varrho[t] = \{x;\ (t,x) \in \varrho \text{ for some } t \in A\}$$

5.2 LEMMA. For every closed set A, the image $\varrho[A]$ is closed, and the map $A \mapsto \varrho[A] : \mathcal{C}(X) \to \mathcal{C}(Y)$ is u.s.c.

In the special case, where $\varepsilon \subseteq X \times X$ is a closed entourage of the uniform structure on X, the set $\varepsilon[A]$ is the ε-neighborhood of X and we have

5.3 COROLLARY. Let ε be a closed entourage of X. Assigning to every closed set $A \subseteq X$ its ε-neighborhood $\varepsilon[A]$ is an u.s.c. function on $\mathcal{C}(X)$.

Let $\mathcal{K}$ be a continuous alignment on X. As the convex closure operator $c_{\mathcal{K}} : \mathcal{C}(X) \to \mathcal{K}$ is continuous, the composition $A \mapsto c_{\mathcal{K}}\varepsilon[A] : \mathcal{C}(X) \to \mathcal{K}$ is also u.s.c. Let us denote by

$$\varepsilon(A) = c_{\mathcal{K}}\, \varepsilon[A]$$

the convex closed hull of the ε-neighborhood $\varepsilon[A]$. For arbitrary closed sets A, the convex closure $\varepsilon(A)$ need not to be small. But for convex set we have:

5.4 LEMMA. Let $\mathcal{B}$ be a base of closed entourages ε for the uniform structure on X. Then for every $\varepsilon \in \mathcal{B}$

$$K \mapsto \varepsilon(K) : \mathcal{K} \to \mathcal{K}$$

is u.s.c. and for all $K \in \mathcal{K}$:

$$K = \bigcap_{\varepsilon \in \mathcal{B}} \varepsilon(K)$$

Proof. We just have to verify the last claim. Thus, let $x \notin K \in \mathcal{K}$. By regularity, there is convex neighborhood $L \in \mathcal{K}$ not containing x. As K is compact, there is an $\varepsilon \in \mathcal{B}$ such that the ε-neighborhood $\varepsilon[A]$ is contained in L. As L is convex, the closed convex hull $\varepsilon(A) = c\ell'_{\mathcal{K}}\,\varepsilon[A]$ is also contained in L. Thus $x \notin \bigcap_{\varepsilon \in \mathcal{B}} \varepsilon(A)$.

Proof of 5.1: Let $F : X \to \mathcal{K}$ be u.s.c. We fix a base $\mathcal{B}$ of closed symmetric entourages ε for the uniform structure of X, and we approximate F by the the functions

$$\varepsilon F : x \mapsto F(x) \mapsto \varepsilon(F(x)) : X \to \mathcal{K}$$

which are also u.s.c. by the preceeding lemma. We shall see that the (approximate) fixed point set of εF:

$$G_\varepsilon = \{x \in X ; x \in \varepsilon(F(x))\}$$

is not empty. As εF is u.s.c, G_ε is closed. As $\varepsilon_1 \subseteq \varepsilon_2$ implies $G_{\varepsilon_1} \subseteq G_{\varepsilon_2}$, the family G_ε is nested when ε runs through $\mathcal{B}$. It follows that $G := \bigcap_{\varepsilon \in \mathcal{B}} G_\varepsilon$ is non-empty. Every $x \in G$ is a fixed point of F: Indeed, if $x \in G$, then $x \in \bigcap_{\varepsilon \in \mathcal{B}} \varepsilon F(x) = F(x)$ by the preceeding lemma. It remains to prove that $G_\varepsilon \neq \emptyset$ for a given closed symmetric entourage ε: By the compactness of X, there are finitely many points $x_1, \ldots, x_n$ such that X is covered by their ε-neighborhoods $\varepsilon[x_i]$. Let P be the $\mathcal{K}$-polygon generated by $x_1, \ldots, x_n$. As every point of X is ε-close to P and as ε is symmetric, $\varepsilon(F(x)) \cap P \neq \emptyset$ for all. Thus, $x \mapsto \varepsilon(F(x)) \cap P$ is a u.s.c. function from P into $\mathcal{K}|P$. As by the hypothesis polygons have the fixed point property, εF has a fixed point in P, i.e. $G_\varepsilon \neq \emptyset$.

5.5 **COMPATIBLE ENTOURAGES.** Given a collection $\mathcal{K}$ of non-empty closed subsets of a compact Hausdorff space X, a closed relation $\rho \subseteq K \times K$ is said to be compatible with $\mathcal{K}$, if $\rho[K] \in \mathcal{K}$ for every $K \in \mathcal{K}$.

Thus, a closed entourage ε is compatible with $\mathcal{K}$ iff the ε-neighborhood $\varepsilon[K]$ of every $K \in \mathcal{K}$ belongs to $\mathcal{K}$. If X has a basis $\mathcal{B}$ of closed symmetric entourages ε which are compatible with X, then every $K \in \mathcal{K}$ has a neighborhood basis belonging to $\mathcal{K}$, i.e. $\mathcal{K}$ satisfies the regularity condition (R), and we have:

PROPOSITION. If $\mathcal{K}$ is a closure system of non-empty closed subsets of a compact Hausdorff space X such that there is a base of closed symmetric $\mathcal{K}$-compatible entourages for the uniform structure of X, then $\mathcal{K}$ is a continuous convexity.

From M. Van de Vel I have learned that a continuous convexity need not admit a base of closed symmetric compatible entourages. Thus, our result is stricly more general than Wieczorek's theorem 5.2 [Wk]. Indeed, for a closure system $\mathcal{K}$ of non-empty closed subsets of a compact Hausdorff space X, the following are equivalent:

(1) $c_{\mathcal{K}} : \mathcal{C}(X) \to \mathcal{C}(X)$ is (uniformly) continuous w.r.t. λ.

(2) $\mathcal{K}$ is λ-closed in $\mathcal{C}(X)$ and regular.

(3) For every entourage ε of X there is an entourage δ such that $c_{\mathcal{K}}\, \delta[C] \subseteq \varepsilon[C]$

5.6 **CONSEQUENCES.** Clearly, the proof of 5.1 is modelled after Ky Fan's and Glicksberg's extension of Kakutani's fixed point theorem from the finite dimensional to the infinite dimensional case. Of course, the Ky Fan and Glicksberg proof is contained in ours when specialised to compact convex sets.

We also may apply our result 5.1 to obtain Wallaces fixed point theorem for trees (see 2.7b): A u.s.c. map f from a tree X into its set $\mathrm{Conn}(X)$ of subcontinua has a fixed point. A "polygon" in this setting is a tree with finitely many branches and theorem 5.1 reduces the fixed point problem to such "finite" trees.

5.7 COMPRESSIBLE SPACES. According Wieczorek [Wk] a compact space X with a continuous convexity $\mathcal{K}$ is called *compressible* if the following holds: Whenever $x_1, \dots, x_n$ is a sequence of points in X and $\mathcal{U}_1, \dots, \mathcal{U}_n$ an open covering of X, then there is a continuous function $\varphi: X \to X$ such that $\varphi(x) \in c_{\mathcal{K}}\{x_i ; x \in A_i\}$ for all $x \in X$.

The following is a simple adaption of Wieczorek's 5.1 to our setting:

5.8 *Let X be a compact space with a compressible continuous convexity $\mathcal{K}$. If every single-valued map $\varphi: X \to X$ has a fixed point, then every multivalued u.s.c map $f: X \to \mathcal{K}$ has a fixed point, too.*

Proof. For every open symmetric entourage ε of X choose finitely many points $y_1^\varepsilon, \dots, y_n^\varepsilon$ whose ε-neighborhoods $\varepsilon[y_i^\varepsilon]$ cover X. Choose $x_i^\varepsilon \in f[\varepsilon[y_i^\varepsilon]] = f\circ\varepsilon[y_i^\varepsilon]$ arbitrarily. By the compressibility, there is a continuous function $\varphi^\varepsilon: X \to X$ such that $\varphi^\varepsilon(x) \in c_{\mathcal{K}}\{x_i^\varepsilon ; x \in \varepsilon[y_i^\varepsilon]\}$. By hypothesis, φ^ε has a fixedpoint x^ε. In the compact space X, the net x^ε (where ε runs through all open entourages ε) has a cluster point x_0. We want to show that x_0 is a fixed point of f, i.e. $x_0 \in f(x_0)$. For this, it suffices to prove that $x_0 \in K$ for every $K \in \mathcal{K}$ which is a neighborhood of $f(x_0)$. Thus, let $K \in \mathcal{K}$ be a neighborhood of $f(x_0)$. As f is u.s.c, there is a neighborhood $\mathcal{U}$ of x_0 such that $f[\mathcal{U}] \subseteq K$. Let ε' be an open symmetric entourage such that $\varepsilon'\circ\varepsilon'\circ\varepsilon'(x_0) \subseteq \mathcal{U}$. For any open entourage $\varepsilon \subseteq \varepsilon'$, one also has $\varepsilon\circ\varepsilon\circ\varepsilon'[x_0] \subseteq \mathcal{U}$, i.e. $y \in \varepsilon'[x_0]$ implies $\varepsilon\circ\varepsilon[y] \subseteq \mathcal{U}$. Thus, if $y \in \varepsilon'[x_0]$ and $y \in \varepsilon[y_i^\varepsilon]$, i.e. $y_i^\varepsilon \in \varepsilon[y]$,

we have $\varepsilon[y_i^\varepsilon] \subseteq \varepsilon\circ\varepsilon[y] \subseteq \varepsilon\circ\varepsilon\circ\varepsilon'[x_0] \subseteq \mathcal{U}$. For the corresponding x_i^ε we have $x_i^\varepsilon \in f[\varepsilon[y_i^\varepsilon]] \subseteq f[\mathcal{U}] \subseteq K$. We conclude that $\varphi^\varepsilon[y] \in c_{\mathcal{K}}\{x_i^\varepsilon ; y \in \varepsilon[y_i^\varepsilon]\} \subseteq K$ for all $y \in \varepsilon'[x_0]$.
As x_0 is a cluster point of the x^ε, $\varepsilon \subseteq \varepsilon'$, there is some $\varepsilon'' \subseteq \varepsilon'$ such that $x^{\varepsilon''} \in \varepsilon'[x_0]$. We conclude that (letting $y = x^{\varepsilon''}$) $x^{\varepsilon''} = \varphi^{\varepsilon''}(x^{\varepsilon''}) \in K$ from the above. As K is closed, we finally conclude that $x_0 \in K$.

Complements on Topologies on $\mathcal{K}$

Let X be a compact Hausdorff space, $\mathcal{C}$ the collection of all non-empty closed subsets and $\mathcal{K} \subseteq \mathcal{C}$.

On $\mathcal{C}$ we have the

a) the lower topology ν: the closed sets are generated by

$$\downarrow C = \{D \in \mathcal{C};\ D \subseteq C\}, \quad C \in \mathcal{C}$$

b) the Scott topology σ: the open sets are generated by

$$\downarrow \mathcal{U} = \{D \in \mathcal{C};\ D \subseteq \mathcal{U}\}, \quad \mathcal{U} \text{ open } \subseteq X \quad {}^{*)}$$

the closed sets are generated by

$$\langle C, X\rangle = \{D \in \mathcal{C};\ D \cap C \neq \emptyset\}$$

On $\mathcal{K}$ we have

a) the lower topology $\nu_{\mathcal{K}}$: the closed sets are generated by

$$\downarrow K = \{L \in \mathcal{K};\ L \subseteq K\}, \quad K \in \mathcal{K}$$

Clearly $\nu_{\mathcal{K}}$ is coarser than $\nu|\mathcal{K}$. If $\mathcal{K}$ is a closure system, then $\nu_{\mathcal{K}}$ is the quotient topology of ν via the closure operator $c_{\mathcal{K}}: \mathcal{C} \to \mathcal{K}$

b) Question: When is $\nu_{\mathcal{K}} = \nu|\mathcal{K}$ (Conjecture: If $\mathcal{K}$ closed in $\mathcal{C}$)

b_1) the Scott topology $\sigma_{\mathcal{K}}$: the open sets are generated by

$$\Downarrow K = \{L \in \mathcal{K};\ L \subseteq K^{\circ}\}, \quad K \in \mathcal{K}$$

Clearly $\sigma_{\mathcal{K}}$ is coarser than $\sigma|\mathcal{K}$. If $\mathcal{K}$ is regular, then $\sigma_{\mathcal{K}} = \sigma|\mathcal{K}$

b_2) the weak topology ω: The closed sets are generated by

$$\langle K, X\rangle_{\mathcal{K}} = \{L \in \mathcal{K};\ L \cap K \neq \emptyset\}.$$

The open sets are generated by

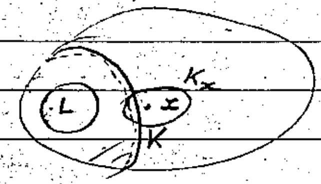
$$\downarrow(X \setminus K) = \{L \in \mathcal{K};\ L \subseteq X \setminus K\}, \quad K \in \mathcal{K}$$

*) One may replace $\downarrow \mathcal{U}$ by $\Downarrow C = \{D \in \mathcal{C};\ D \subseteq C^{\circ}\}$, $C \in \mathcal{C}$.

ω is coarser than $\sigma|\mathcal{K}$. In general, I do not know about the relation between ω and $\sigma_{\mathcal{K}}$. In the regular case we have $\sigma_{\mathcal{K}} = \sigma|\mathcal{K}$, whence ω is coarser than $\sigma_{\mathcal{K}}$. The following simple example shows that ω may be strictly coarser than $\sigma_{\mathcal{K}}$:

Example. Let $X = [0,1]$, $\mathcal{K} = \{[0,a];\ 0 \leq a \leq 1\}$.
For $K \in \mathcal{K}$, the set $\langle K, X\rangle_{\mathcal{K}} = \{L \in \mathcal{K};\ L \cap K \neq \emptyset\} = \mathcal{K}$.
Thus, ω is the indiscrete topology; but $\sigma_{\mathcal{K}}$ has as open sets all sets of the form $\{[0,b];\ b < a\}$, $0 \leq a \leq 1\}$

But: If $\mathcal{K}$ is regular and T_1, then $\omega = \sigma_{\mathcal{K}}$.
(Proof: Let $K \in \mathcal{K}$ be given. Let $L \subseteq K^{\circ}$ and show that there is a ω-open set A such that $L \in A \subset \Downarrow K$.



For $x \notin K^{\circ}$, let $K_x \in \mathcal{K}$ be a neighborhood of x that does not meet L. Finitely many of them, say $K_{x_1}^{\circ}, \ldots, K_{x_n}^{\circ}$ cover $X \setminus K^{\circ}$. Thus the intersection A of the ω-open subbasic sets $\downarrow(X \setminus K_{x_1}), \ldots, \downarrow(X \setminus K_{x_n})$ is contained in $\Downarrow K$ and has L as a member.)

b_3) the convex-open topology γ: the open sets are generated by

$$\downarrow \mathcal{U} = \{L \in \mathcal{K};\ L \subseteq \mathcal{U}\}, \quad \mathcal{U} \text{ convex open}$$

Clearly γ is coarser than $\sigma|\mathcal{K}$. If every $K \in \mathcal{K}$ has a neighborhoodbasis of open convex sets, then $\gamma = \sigma|\mathcal{K}$, and conversely.

$\boxed{\text{SUMMARY}}$ The Lawson topology $\lambda_{\mathcal{K}}$ on $\mathcal{K}$ is defined to be $\sigma_{\mathcal{K}} \vee \nu_{\mathcal{K}}$.
We have

$$\begin{aligned}
\lambda_{\mathcal{K}} &= \nu_{\mathcal{K}} \vee \sigma|_{\mathcal{K}} && \text{if } \mathcal{K} \text{ is regular} \\
&= \nu_{\mathcal{K}} \vee \omega && \text{if } \mathcal{K} \text{ is regular } T_1 \\
&= \nu_{\mathcal{K}} \vee \gamma && \text{if } \mathcal{K} \text{ is (strongly) locally convex} \\
&= \nu|_{\mathcal{K}} \vee \sigma|\mathcal{K} = \lambda|\mathcal{K} && \text{if } \mathcal{K} \text{ is a regular closure system } \lambda \text{ closed in } \mathcal{C}.
\end{aligned}$$