SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)	
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At the Bremen Conference, Alan Day mentioned that semilattices also have injective hulls and that there may be a connection between the Zariski topology on semilattices and essential extensions. And indeed, he was right, and we will discuss this matter in these notes.

1. Injective Hulls of Semilattices.

Let S be a semilattice. We say that S is *injective*, if it is injective in the category of all semilattices with semilattice homomorphisms as morphisms. The following definitions and results are taken from [5] and [1]:

1.1. Proposition. A semilattice S is injective if and only if S is a complete Brouwerian lattice (i.e. a distributive complete meet-continuous lattice). []

Let S and T be semilattice and let $i : S \to T$ be an embedding. If for every semilattice homomorphism $g : T \to L$ the composition $g \circ i$ is injective if and only if g is injective, then T is called an essential extension of S. Every semilattice S admits a maximal essential extension and this maximal essential extension happens to be injective. We will denote this maximal essential extension by $\mathcal{D}(S)$; this semilattice $\mathcal{D}(S)$ is called the injective hull of S. The injective hull of a semilattice S can be constructed in the following way:

A subset $A \subseteq S$ is called *admissible* if and only if the following two conditions hold:

(a) The supremum $\sup A$ exists in S.

(b) For every element $s \in S$ the supremum $\sup(A \wedge s)$ exists in S and we have $\sup(A \wedge s) = s \wedge \sup A$.

A subset $M \subseteq S$ is called a *D-ideal* if and only if

(c) M is a lower set.

(d) If $A \subseteq M$ is an admissible subset, then $\sup A \in M$.

Let $\mathcal{D}(S)$ be the set of all D-ideals of S. Then $\mathcal{D}(S)$ is a complete lattice (where the infima agree with set theoretical intersections); this lattice turns out be be Brouwerian.

1.2. Proposition. Under the canonical embedding

$$i_S : S \to \mathcal{D}(S)$$
$$s \mapsto \downarrow s$$

the semilattice $\mathcal{D}(S)$ becomes the injective hull of S. Moreover, the embedding $f : S \to T$ is an essential embedding if and only if there is an embedding $g : T \to \mathcal{D}(S)$ such that $i_S = g \circ f$. ▯

2. Essential Extensions and the Zariski-Topology.

Let S be a semilattice and let p and q be semilattice polynomials in one variable with constants from S. We define

$$[p = q]_S = \{x \in S \mid p(x) = q(x)\}.$$

The ***Zariski-topology*** (Z-topology) on S is the coarsest topology making all sets of the form $[p = q]_S$ closed, where p and q ranges over all semilattice polynomials in one variable with constants from S. Recall from [3] that the following sets form a subbase for the closed sets of the Z-topology:

$$[a \wedge x \leq b]_S = \{x \in S \mid a \wedge x \leq b\},$$
$$\uparrow a = \{x \in S \mid a \leq x\}.$$

2.1. Proposition. Let S be a semilattice and let $a, b \in S$ be given. Then $[a \wedge x \leq b]_S$ is a D-ideal, i.e. we have $[a \wedge x \leq b] \in \mathcal{D}(S)$.

Proof. Let $M \subseteq [a \wedge x \leq b]_S$ be an admissible set. Then for every $m \in M$ we have $a \wedge m \leq b$. Since M is admissible, this implies $a \wedge \sup M = \sup(M \wedge a) \leq b$, i.e. $\sup M \in [a \wedge x \leq b]_S$. ▯

Next, we show that conversely every D-ideal is closed in the Z-topology. We will even show a stronger result, namely that the sets of the form $[a \wedge x \leq b]_S$ order generate $\mathcal{D}(S)$. We will start with a lemma:

2.2. Lemma. Let S be a semilattice and let $M \subseteq S$ be D-ideal. If $x_0 \in S \setminus M$, then there are elements $a, b \in S$ such that for all elements $m \in M$ we have $m \wedge a \leq b < a \leq x_0$.

Proof. Since $x_0 \notin M$, and since M is a D-ideal, either the set $\downarrow x_0 \cap M$ is not admissible or this set has not supremum x_0. In the first case, there is an element $b_0 \in S$ such that $b_0 \wedge (\downarrow x_0 \cap M) = \downarrow(b_0 \wedge x_0) \cap M$ has not $b_0 \wedge x_0$ has its supremum. In the second case, we can find such an element b_0, too, namely $x_0 = b_0$. Hence in either case we can state:

> There are elements $b_0, b \in S$ such that for all elements $m \in M$ we have $m \wedge b_0 \wedge x_0 \leq b < b_0 \wedge x_0$.

Hence, the elements b and $a = b_0 \wedge x_0$ have the required properties. ▯

2.3. Proposition. Let S be a semilattice and let $M \subseteq S$ be a D-ideal. If $x_0 \in S \setminus M$, then there are elements $a, b \in S$ such that $x_0 \notin [a \wedge x \leq b]_S$ and $M \subseteq [a \wedge x \leq b]_S$.

Proof. If we pick the elements $a, b \in S$ as stated in Lemma (2.2), then $a \leq x_0$ implies $a \wedge x_0 = a > b$, hence $x_0 \notin [a \wedge x \leq b]_S$. Clearly, $a \wedge x \leq b$ for all $m \in M$ implies $M \subseteq [a \wedge x \leq b]_S$. ▯

2.4. Proposition. Let S be a semilattice. Then the D-ideals of S are exactly the intersections of sets of the form $[a \wedge x \leq b]_S$, $a, b \in S$. Especially, all D-ideals are closed lower sets in the Zariski-topology. ▯

For the following result, note that the Z-topology on every semilattice which happens to be a Brouwerian lattice is equal to the interval topology.

2.5. Theorem. The embedding $i_S : S \to \mathcal{D}(S)$ is a topological embedding for the Z-topologies on S and $\mathcal{D}(S)$.

Proof. Since $\mathcal{D}(S)$ is a Brouwerian lattice, the Z-topology on $\mathcal{D}(S)$ is the interval topology.

First of all, we show that the mapping i_S is continuous. Thus, let $M \in \mathcal{D}(S)$ be any D-ideal of S. Then

$$\begin{aligned} i_S(x) \leq M \quad &\Leftrightarrow \downarrow x \subseteq M \\ &\Leftrightarrow x \in M, \end{aligned} \tag{1}$$

hence $i_S^{-1}(\{N \in \mathcal{D}(S) \mid N \subseteq M\}) = M$, and this set is closed by Proposition (2.4). Moreover,

$$\begin{aligned} i_S(x) \geq M \quad &\Leftrightarrow M \subseteq \downarrow x \\ &\Leftrightarrow x \in \bigcap\{\uparrow m \mid m \in M\}, \end{aligned} \tag{2}$$

and this set is closed in the Z-topology of S, too.

It remains to check that the embedding i_S is open onto its image. Let $M = [a \wedge x \leq b]_S$. Then, by (1), $M = i_S^{-1}(\downarrow M)$. If $x \in S$ is given, then, by (2), we have $\uparrow x = i_S^{-1}(\{M \in \mathcal{D}(S) \mid \uparrow x \subseteq M\})$. Hence the relative topology on S inherited from the interval topology on $\mathcal{D}(S)$ is as least as fine as the Z-topology. []

From this last result and from proposition (1.2) we conclude:

2.6. Theorem. Let S and T be semilattices and let $f : S \to T$ be an essential embedding. Then f is a topological embedding for the Z-topologies on S and T, respectively. []

We now study some properties of the closure operator in the Z-topology:

2.7. Proposition. Let S be a semilattice and let $A \subseteq S$. Then A is admissible and $x = \sup_S A$ if and only if $i_S(x) = \sup_{\mathcal{D}(S)} i_S(A)$.

The *proof* of (2.7) follows immediately from the meet-continuity of $\mathcal{D}(S)$ and the definition of admissibility. []

2.8. Proposition. Let S be a semilattice and let $A \subseteq S$ be a lower set. Then the closure of A in the Z-topology is a lower set, too.

Proof. This statement is true for meet-continuous lattices in the interval topology and hence for arbitrary semilattices by (2.5). []

For the next result, recall that an ideal of a semilattice is an directed lower set. The set of all ideals of S will be denoted by $Id(S)$. Note that $Id(S)$ is a semilattice under $\cap$ (see [2, p.6, Exercise 1.15]). Also, in order to avoid confusion, we will denote suprema taken in S by $\sup_S$ and suprema taken in $\mathcal{D}(S)$ by $\sup_{\mathcal{D}(S)}$.

2.9. Proposition. Let S be a semilattice and let $I \subseteq S$ be an ideal of S. Then the closure $\bar{I}$ of I in the Z-topology is given by

$$\bar{I} = \{x \in S \mid \downarrow x \cap I \text{ is admissible and has supremum } x\}.$$

Moreover, $\bar{I}$ is a D-ideal.

Proof. Let $i_S(I)$ be the image of I in $\mathcal{D}(S)$. Furthermore, let $i_S(I)^-$ be the closure of $i_S(I)$ in the Z-topology of $\mathcal{D}(S)$. Then $\bar{I} = i_S^{-1}(i_S(I)^-)$ by (2.5). Let $A = \sup_{\mathcal{D}(S)} i_S(I) \in \mathcal{D}(S)$. Since on $\mathcal{D}(S)$ the Z-topology and the interval topology agree, $\downarrow A$ is closed in the Z-topology on $\mathcal{D}(S)$. Hence, we conclude that

$$i_S(I)^- \subseteq \downarrow A.$$

Now let $x \in \bar{I}$. Then

$$i_S(x) = \downarrow x \subseteq A,$$

therefore the meet-continuity of $\mathcal{D}(S)$ yields

$$\begin{aligned}\downarrow x &= \sup{}_{\mathcal{D}(S)}(\downarrow x \cap i_S(I)) \\ &= \sup{}_{\mathcal{D}(S)}(i_S(\downarrow x \cap I)).\end{aligned}$$

From (2.7) we obtain that $\downarrow x \cap I$ is admissible and has supremum x. This verifies

$$\bar{I} \subseteq \{x \in S \mid \downarrow x \cap I \text{ is admissible and has supremum } x\}.$$

Conversely, assume that $\downarrow x \cap I$ is admissible and has supremum x. Using (2.7) again, we obtain

$$\downarrow x = \sup{}_{\mathcal{D}(S)}(i_S(\downarrow x \cap I)).$$

Since the intersection of two ideals is again an ideal, $i_S(\downarrow x \cap I)$ is directed and hence converges to its supremum in the interval topology. Hence $\downarrow x \cap I$ converges to x in the Z-topology by (2.5). This proves the other inclusion.

It remains to show that $\bar{I}$ is a $\mathcal{D}$-ideal. First, note that $\bar{I}$ is a lower set by (2.8). Let $M \subseteq I$ be an admissible subset and let $x = \sup_S M$. We have to show that $x \in \bar{I}$. By (2.7), $i_S(x) = \sup_{\mathcal{D}(S)} i_S(M) \subseteq \sup_{\mathcal{D}(S)} i_S(I)$ and therefore $i_S(x) = \sup_{\mathcal{D}(S)}(i_S(x) \cap i_S(I))$ by the meet-continuity of $\mathcal{D}(S)$. Proposition (2.7) now yields that $\downarrow x \cap I$ is admissible and has supremum x. Thus $x \in \bar{I}$ by the part of the proposition we already proved. ▯

For the following result, recall from (2.4) that every element $A \in \mathcal{D}(S)$ is a subset $A \subseteq S$ which is closed in the Z-topology.

2.10. Proposition. Let S be a semilattice.

(i) For every ideal $I \in Id(S)$ we have $\bar{I} = \sup\{i_S(x) \mid x \in I\}$.

(ii) The mapping

$$\begin{aligned}{}^- : Id(S) &\to \mathcal{D}(S) \\ I &\mapsto \bar{I}\end{aligned}$$

preserves finite intersections, i.e. is a semilattice homomorphism.

Proof. (i): Let $A = \sup_{D(S)} i_S(I) \in D(S)$. We have to show that $A = \bar{I}$. We already saw in the proof of (2.9) that $\downarrow x \subseteq A$ for every $x \in \bar{I}$. This shows $\bar{I} \subseteq A$. Conversely, let $x \in A$. Then $\downarrow x \subseteq A$ and hence the same arguments as the once used in the proof of (2.9) show that $\downarrow x \cap I$ is admissible and has supremum x. By (2.9), this shows that $x \in \bar{I}$. This verifies the inclusion $A \subseteq \bar{I}$.

(ii) Let I and J be two ideals of S. Then the meet-continuity of $D(S)$ and the fact that i_S preserves finite infima yield

$$\begin{aligned}
\sup_{D(S)}(i_S(I)) \cap \sup_{D(S)}(i_S(J)) &= \sup_{D(S)}\{i_S(x) \mid x \in I\} \cap \sup_{D(S)}\{i_S(y) \mid y \in J\} \\
&= \sup_{D(S)}\{i_S(x \wedge y) \mid x \in I, y \in J\} \\
&= \sup_{D(S)}\{i_S(z) \mid z \in I \cap J\} \\
&= \sup_{D(S)}(i_S(I \cap J)).
\end{aligned}$$

Therefore, (ii) is a consequence of (i). ∎

The proof of this proposition would be much easier if we could assure that the closure of an ideal is again an ideal. However, this is not the case as the following example shows: example shows:

2.11. Example. Consider the following subsemilattice S of the unit square $[0,1] \times [0,1]$:

$$S = \{(x,y) \in [0,1] \times [0,1] \mid x = 0 \text{ or } y = 0 \text{ or } x = y < 1\}.$$

Then the unit square is an essential extension of S, hence the Z-topology on S agrees with the ordinary Euclidean topology. Moreover,

$$I = S \setminus \{(1,0),(0,1)\}$$

is a directed lower set of S, hence an ideal. Note that I is dense in S, i.e. $\bar{I} = S$. However, S itself is not directed (the elements $(1,0)$ and $(0,1)$ do not have an upper bound in S). Therefore, S is not an ideal of S. This shows that the closure of an ideal does not have to be an ideal.

3. Semilattices for which the Z-Topology is Hausdorff.

For distributive lattices L, the lattice version of the Z-topology is Hausdorff if and only if L admits an essential extension which is completely distributive. We will prove a similar result here for semilattices. We will show that for a semilattice S, the Z-topology on S is Hausdorff if and only if S admits an essential extension $\rho(S)$

which is a continuous semilattice (i.e. a continuous lattice without largest element 1) such that on $\rho(S)$ the Lawson topology and the Z-topology agree. Examples for such semilattices are semilattices of finite breadth (see Theorem (3.10)) and hypercontinuous lattices (see Theorem (3.11)).

3.1. Definition. Let S be a semilattice. By $\rho(S)$ we denote the smallest subsemilattice $\rho(S) \subseteq \mathcal{D}(S)$ such that

(1) $i_S(S) \subseteq \rho(S)$.

(2) $\rho(S)$ is the smallest subsemilattice of $\mathcal{D}(S)$ which satisfies (1) and at the same time is closed in $\mathcal{D}(S)$ under the formation of directed suprema and arbitrary non-empty infima. ▯

Note that $\rho(S)$ is always an essential extension of S. It is clear that $\rho(S)$ is always a meet-continuous complete semilattice (i.e. infima of non-empty sets and suprema of directed sets always exist). We will now show that the Z-topology on $\rho(S)$ is Hausdorff if and only if the Z-topology on S is Hausdorff.

3.2. Proposition. Let S be a semilattice. Then $\rho(S)$ is the smallest subset of $\mathcal{D}(S)$ which is closed under the formation of directed suprema and (non-empty) filtered infima.

Proof. Since $\mathcal{D}(S)$ is meet-continuous and since $i_S(S)$ is closed under finite infima, the smallest subset containing $i_S(S)$ closed under the formation of directed suprema and (non-empty) filtered infima is a subsemilattice again and hence agrees with $\rho(S)$. ▯

3.3. Proposition. Let S be a semilattice. Then the Z-topology on S is Hausdorff if and only if for every pair of elements $x_0, y_0 \in S$ such that $x_0 < y_0$ there is a finite set $F \subseteq F$ and finitely many elements $a_1, b_1, \ldots, a_n, b_n \in S$ such that

(1) $x_0 \notin \uparrow F$.

(2) $y_0 \notin [a_1 \wedge x \leq b_1]_S \cup \ldots \cup [a_n \wedge x \leq b_n]_S$.

(3) $S = \uparrow F \cup [a_1 \wedge x \leq b_1]_S \cup \ldots \cup [a_n \wedge x \leq b_n]_S$.

Proof. Assume first of all that the Z-topology is Hausdorff and let $x_0, y_0 \in S$ be given such that $x_0 < y_0$. Then we can separate x_0 and y_0 by disjoint open neighborhoods. Translating this into closed subsets of S and taking into account that set of the form $\uparrow a$ and $[a \wedge x \leq b]_S$ form a subbase for the closed sets, we therefore can find elements $a_1, b_1, \ldots, a_m, b_m, \ldots a_n, b_n \in S$ and finite subset $F_1, F_2 \subseteq S$ such that

(a) $x_0 \notin [a_1 \wedge x \leq b_1]_S \cup \ldots \cup [a_m \wedge x \leq b_m]_S \cup \uparrow F_1$.

(b) $y_0 \notin [a_{m+1} \wedge x \leq b_{m+1}]_S \cup \ldots \cup [a_n \wedge x \leq b_n]_S \cup \uparrow F_2$.
(c) $S = \uparrow F_1 \cup \uparrow F_2 \cup [a_1 \wedge x \leq b_1]_S \cup \ldots \cup [a_n \wedge x \leq b_n]_S$.

Since $x_0 < y_0$, the elements $a_1, b_1, \ldots, a_n, b_n \in S$ and the finite set $F = F_1 \cup F_2 \subseteq S$ satisfy (1) - (3) of the proposition.

In order to prove the converse, let $x_1, y_1 \in S$ be given and assume that $x_1 \neq y_1$. Without loss of generality we may assume that $x_1 \not\leq y_1$. Let $y_0 := y_1$ and let $x_0 = x_1 \wedge y_1$. Then $x_0 < y_0$ and hence we may find a finite set $F \subseteq S$ and elements $a_1, b_1, \ldots, a_n, b_n \in S$ such that (1) - (3) hold. Then, since sets of the form $[a \wedge x \leq b]_S$ are lower sets and since $x_0 \leq x_1$, properties (1) - (3) also hold with x_1 and y_1 instead of x_0 and y_0. Hence

$$U = S \setminus ([a_1 \wedge x \leq b_1]_S \cup \ldots \cup [a_n \wedge x \leq b_n]_S)$$

and

$$V = S \setminus \uparrow F$$

are disjoint neighborhoods of x_1 and y_1 respectively. ∎

3.4. Theorem. Let S be a semilattice. Then the Z-topology on S is Hausdorff if and only if the Z-topology on $\rho(S)$ is Hausdorff.

Proof. If the Z-topology on $\rho(S)$ is Hausdorff, then the Z-topology on S is Hausdorff by (2.6).

Conversely, assume that the Z-topology on S is Hausdorff. Let $X_0, Y_0 \in \rho(S)$ and assume that $X_0 \subseteq Y_0, X_0 \neq Y_0$. We would like to verify (3.2) with the elements X_0 and Y_0. Since $X_0, Y_0 \subseteq S$ are D-ideals, this means that there is an element $y_1 \in S$ such that $y_1 \in Y_0 \setminus X_0$. By Lemma (2.2) we can find elements $a, b \in S$ such that

$$m \wedge a \leq b < a \leq y_1 \qquad \text{for all } m \in X_0.$$

Since the Z-topology on S is Hausdorff, we can apply (3.2) in order to find a finite subset $F \subseteq S$ and elements $a_1, b_1, \ldots a_n, b_n \in S$ such that

(1) $b \notin \uparrow F$,
(2) $a \notin [a_1 \wedge x \leq b_1]_S \cup \ldots \cup [a_n \wedge x \leq b_n]_S$,
(3) $S = \uparrow F \cup [a_1 \wedge x \leq b_1]_S \cup \ldots \cup [a_n \wedge x \leq b_n]_S$.

Now let $G = F \setminus X_0$. Then G is still a finite set. Moreover, let

$$a_i' = a_i \wedge a \qquad \text{for all } 1 \leq i \leq n.$$

Then we have

(1′) $b \notin \uparrow G$,
(2′) $a \notin [a \wedge x \leq b]_S \cup [a'_1 \wedge x \leq b_1]_S \cup \ldots \cup [a'_n \wedge x \leq b_n]_S$.
(3′) $S = \uparrow G \cup [a \wedge x \leq b] \cup [a'_1 \leq b_1]_S \cup \ldots \cup [a'_n \leq b_n]_S$.

Here, (1′) and (2′) are obvious. To verify (3′), let $u \in S$. Consider the element $a \wedge u$ and assume that

$$a \wedge u \notin [a \wedge x \leq b]_S \tag{1}$$

and

$$a \wedge u \notin [a'_i \wedge x \leq b_i]_S \qquad \text{for all } 1 \leq i \leq n. \tag{2}$$

Then we conclude that

$$a \wedge u \notin [a_i \wedge x \leq b_i]_S \qquad \text{for all } 1 \leq i \leq n.$$

Since $S = \uparrow F \cup [a_1 \wedge x \leq b_1]_S \cup \ldots \cup [a_n \wedge x \leq b_n]_S$, we conclude that $a \wedge u \in \uparrow F$. We would like to show that in fact $a \wedge u \in \uparrow G$. Let $f \in F$ such that $f \leq a \wedge u$ and assume that $f \in X_0$. By the choice of a and b, this would imply that $a \wedge f \leq b$. Since $f \leq a \wedge u \leq a$, this would imply $f = f \wedge a \leq b$, contradicting $b \notin \uparrow F$.

Therefore, statements (1) and (2) above imply

$$a \wedge u \in \uparrow G.$$

Finally, note that $a \wedge u \in [a \wedge x \leq b]_S$ is equivalent to $u \in [a \wedge x \leq b]_S$ and $a \wedge u \in [a_i \wedge a \wedge x \leq b_i]_S$ is equivalent to $u \in [a_i \wedge a \wedge x \leq b_i]_S$. Hence (3′) follows.

In the next step, we show that

(1″) $X_0 \notin \uparrow i_S(G)$;
(2″) $Y_0 \notin [i_S(a) \cap X \subseteq i_S(b)]_{\rho(S)} \cup [i_S(a'_1) \cap X \subseteq i_S(b_n)]_{\rho(S)} \cup \ldots$
$\ldots \cup [i_S(a'_n) \cap X \subseteq i_S(b_n)]_{\rho(S)}$;
(3″) $\rho(S) = \uparrow i_S(G) \cup [i_S(a) \cap X \subseteq i_S(b)]_{\rho(S)} \cup [i_S(a'_1) \cap X \subseteq i_S(b_n)]_{\rho(S)} \cup$
$\ldots \cup [i_S(a'_n) \cap X \subseteq i_S(b_n)]_{\rho(S)}$.

In order to verify (1″), assume that we had $X_0 \in \uparrow i_S(G)$. Then $i_S(g) \subseteq X_0$ for a certain $g \in G$ and therefore $g \in X_0 \cap G \neq \emptyset$, contradicting the choice of G.

Next, assume that (2″) does not hold, i.e. that

$$Y_0 \in [i_S(a) \cap X \subseteq i_S(b)]_{\rho(S)} \cup [i_S(a'_1) \cap X \subseteq i_S(b_n)]_{\rho(S)} \cup \ldots$$
$$\ldots \cup [i_S(a'_n) \cap X \subseteq i_S(b_n)]_{\rho(S)}.$$

Then either $Y_0 \cap i_S(a) \subseteq i_S(b)$ or $Y_0 \cap i_S(a_i') \subseteq i_S(b_i)$ for some $1 \leq i \leq n$. In the first case, this would imply

$$\begin{aligned} a &\in {\downarrow} y_1 \cap {\downarrow} a \\ &\subseteq Y_0 \cap {\downarrow} a \\ &= Y_0 \cap i_S(a), \\ &\subseteq i_S(b) \\ &= {\downarrow} b \end{aligned}$$

i.e. $a \leq b < a$, a contradiction. Hence there has to be a number $i \in \{1, \ldots, n\}$ such that

$$Y_0 \cap i_S(a_i') \subseteq i_S(b_i).$$

In this case, we obtain

$$\begin{aligned} a \wedge a_i &\in {\downarrow} y_1 \cap {\downarrow} a_i \cap {\downarrow} a \\ &= {\downarrow} y_1 \cap {\downarrow} a_i' \\ &\subseteq Y_0 \cap i_S(a_i') \\ &\subseteq i_S(b_i) \\ &= {\downarrow} b_i, \end{aligned}$$

i.e. $a \in [a_i \wedge x \leq b_i]_S$, contradicting (2). Hence we verified ($2''$).

Finally, we have to verify ($3''$). Let

$$\begin{aligned} \mathcal{A} = {\uparrow} i_S(G) \cup [i_S(a) \cap X \subseteq i_S(b)]_{\rho(S)} \cup [i_S(a_1') \cap X \subseteq i_S(b_n)]_{\rho(S)} \cup \ldots \\ \ldots \cup [i_S(a_n') \cap X \subseteq i_S(b_n)]_{\rho(S)}. \end{aligned}$$

We have to show that $\rho(S) \subseteq \mathcal{A}$. We will do this by using (3.2): Note that the meet-continuity of $\rho(S)$ implies that $\mathcal{A}$ is closed under directed suprema and non-empty filtered infima. Moreover, by ($3'$), the image $i_S(S)$ of S under i_S is contained in $\mathcal{A}(S)$. Hence, by (3.2), $\rho(S) \subseteq \mathcal{A}$.

An application of (3.3) now finishes the proof of the theorem. ∎

If S is meet-continuous itself, then all the sets of the form $[a \wedge x \leq b]_S$ are Scott-closed in S. Hence we can state (see also [3]):

3.5. Proposition. If S is an up-complete meet-continuous semilattice, then the Lawson-topology on S is equal to or finer than the Z-topology. Especially, the Z-topology is quasicompact. ∎

The last proposition immediately yields

3.6. Proposition. Let S be complete meet-continuous semilattice such that the Z-topology is Hausdorff. Then the Z-topology and the Lawson topology agree. Especially, the Z-topology is compact. ▯

The next result follows from (3.4) and (3.6):

3.7. Corollary. Let S be a semilattice and assume that the Z-topology on S is Hausdorff. Then the Z-topology is actually completely regular. Moreover, the continuous semilattice homomorphisms into the unit interval separate the points of S and S is a topological semilattice in the Z-topology. ▯

We will now list some properties which insure that the Z-topology of a semilattice is Hausdorff. The first such property is finite breadth. We need some preparations: If S is a semilattice, then we denote by $Id(S)$ the semilattice of all ideals of S (an ideal being an up-directed lower set of S).

3.8. Proposition. Let S be a semilattice of finite breadth n. Then $Id(S)$ has also breadth n.

Proof. In the case of lattices, this theorem is well known: The ideal lattice of a lattice L is a homomorphic image of sublattice of an ultrapower of L. The fact that the ideal lattice of L has breadth n, too, now follows from model theoretical considerations. I'm sure that (3.8) is also well known in the case where S is merely a semilattice, but since I could not find any references, I will present a proof here:

Since S is a subsemilattice of $Id(S)$ under the embedding $s \mapsto \downarrow s : S \to Id(S)$, the breadth of $Id(S)$ is at least n. Conversely, we have to show that the breadth of $Id(S)$ is at most n. Therefore, let $I_1, \ldots, I_m \in Id(S)$ be an irredundant family of ideals. We have to show that $m \leq n$. In order to do this, we will pick a family of elements $x_i \in I_i$, $1 \leq i \leq m$, which is irredundant.

First, recall that $I_1, \ldots, I_m$ is irredundant if and only if

$$I_1 \cap \ldots \cap I_{i-1} \cap I_{i+1} \cap \ldots I_m \not\subseteq I_i \qquad \text{for every } i \in \{1, \ldots, m\}.$$

Hence for every number $i \in \{1, \ldots, m\}$ we can pick an element

$$a_i \in (I_1 \cap \ldots \cap I_{i-1} \cap I_{i+1} \cap \ldots \cap I_m) \setminus I_i.$$

Then $a_j \in I_i$ whenever $i \neq j$. Since all the I_i's are ideals, hence directed, we can find elements b_i such that

$$a_j \leq b_i \in I_i \quad \text{for every } i \neq j.$$

The elements $b_1, \ldots, b_m$ then form an irredundant set. Indeed, if we had

$$b_1 \wedge \ldots \wedge b_{i-1} \wedge b_{i+1} \wedge \ldots \wedge b_m \leq b_i$$

for some $i \in \{1, \ldots, m\}$, then, since $a_i \leq b_j$ whenever $i \neq j$, we would obtain $a_i \leq b_i \in I_i$, contradicting the choice of a_i. ▯

3.9. Proposition. Let S be a semilattice of finite breadth n. Then $\rho(S)$ also has breadth n.

Proof. For every subset $H \subseteq \rho(S)$ let

$$H^+ = \{\sup D \mid D \subseteq H \text{ is directed }\},$$
$$H^- = \{\inf F \mid F \subseteq H \text{ is filtered }\}.$$

Firstly, we show

(C) If H is a subsemilattice of $\rho(S)$, and if the breadth of H is finite and equal to n, then the breadth of the subsemilattices H^+ and H^- are also equal to n.

Let us consider H^+ first. In this case, we argue as follows: Consider the mapping

$$\sup : Id(H) \to \rho(S)$$
$$I \mapsto \sup_{\rho(S)} I$$

Since $\rho(S)$ is meet-continuous, this mapping preserves finite infima. Moreover, since H and therefore $Id(H)$ have breadth n, the breadth of the image of the semilattice homomorphism is at most n. Since H^+ is the image of this mapping, the breadth of H^+ is no larger than n. On the other hand, H is a subsemilattice of H^+. Therefore the breadth of H^+ is equal to n.

Next, let us consider H^-. Assume that $a_1, \ldots, a_m \in H^-$ are irredundant. We have to show that $m \leq n$. Since the $a_1, \ldots, a_m$ are irredundant, we know that

$$a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_m \not\leq a_i$$

for every $i \in \{1, \ldots, m\}$. Every element a_i is a filtered infimum of elements of H. Therefore, for every $1 \leq i \leq m$ we can pick an element $b_i \in H$ such that $a_i \leq b_i$ and

$$a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_m \not\leq b_i.$$

Now assume that there would be an i such that

$$b_1 \wedge \ldots \wedge b_{i-1} \wedge b_{i+1} \wedge \ldots \wedge b_m \not\leq b_i.$$

Then we also would have

$$a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_m \le b_1 \wedge \ldots \wedge b_{i-1} \wedge b_{i+1} \wedge \ldots \wedge b_m \le b_i,$$

contradicting the choice of the b_i. Hence the elements $b_1, \ldots, b_m \in H$ form an irredundant set. Since the breadth of H is n, this yields $m \le n$ and therefore the breadth of H^- is a most n. The fact that H is a subsemilattice of H^- shows that the breadth of H^- is exactly n.

The next statement is obvious:

(D) If H_i, $i \in I$, is an up-directed set of subsemilattices of $\rho(S)$, and if the breadth of each H_i is equal to n, then the subsemilattice $\bigcup_{i \in I} H_i$ has breadth n, too.

We now define inductively for every ordinal i less than $|\rho(S)| + 1$ semilattices H_i by

$$\begin{aligned} H_0 &= i_S(S); \\ H_{i+1} &= H_i^{-+}; \\ H_i &= \bigcup_{j<i} H_j \quad \text{if } i \text{ is a limit ordinal}. \end{aligned}$$

Then, by (C) and (D), the breadth of all the semilattices H_i is equal to n. For a certain ordinal i we have finally to arrive at $H_i = H_{i+1}$, which means that H_i is closed under directed suprema and filtered infima. For this i we have $H_i = \rho(S)$ by (3.2). Thus, $\rho(S)$ has breadth n. ▯

We now can conclude

3.10. Theorem. If S is a semilattice of finite breadth, then the Z-topology on S is Hausdorff.

Proof. The semilattice $\rho(S)$ has finite breadth by (3.9). Hence, by [3], the Z-topology and the Lawson-topology agree on $\rho(L)$. Since a meet-continuous lattice of finite breadth is actually continuous by [6], the Z-topology on $\rho(S)$ is Hausdorff. Hence S is Hausdorff in its Z-topology by (3.5). ▯

3.11. Theorem. If S is a semilattice which admits an essential extension which is hypercontinuous, then the Z-topology on S is Hausdorff.

Proof. On a hypercontinuous lattice, the Lawson topology agrees with the interval topology. Since the interval topology is always finer than or equal to the Z-topology (note that $\downarrow a = [a \wedge x = x]_S$), this implies that the Z-topology is Hausdorff. Theorem (3.11) now follows from (3.4). ∎

The last two results suggest that it may be worthwhile to study continuous lattices for which the Z-topology is Hausdorff and hence agrees with the Lawson-topology. In order to have a preliminary name for those continuous lattices, let us call a semilattice S *strongly continuous* if

(1) S is meet-continuous, up-complete and complete.
(2) The Z-topology on S is Hausdorff.

Clearly, every such strongly continuous lattice is continuous. From (3.10) and (3.11) we may deduce that every hypercontinuous lattice and every continuous lattice of finite breadth is strongly continuous. Moreover, if S is a meet-continuous distributive lattice, then the Z-topology and the interval topology agree. Hence a distributive lattice is strongly continuous if and only if it hypercontinuous. What else can we say? Is there a equational characterization of strongly continuous lattices? Do these strongly continuous lattices in some sense form 'one-point compactifications' of semilattices for which the Z-topology is Hausdorff in the same sense as completely distributive lattices do for distributive lattices (see [4])?

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