Proof of a theorem of B.B.

Let L be a continuous lattice s.th. $\ll$ is multiplicative.

(1) The Scott open filters of L form a sublattice of the lattice of all filters of L.
(Let F_1 and F_2 be Scott open filters. $F_1 \cap F_2$ is a Scott open filter in any case. By the multiplicativity of $\ll$, the set

$$F_1 \sqcup F_2 = \uparrow\{x_1 \wedge x_2;\ x_1 \in F_1 \text{ and } x_2 \in F_2\}$$

is not only a filter but also Scott open.

(2) If F_1 and F_2 are maximal Scott open filters, then there are $x_1 \in F_1$, $x_2 \in F_2$ such that $x_1 \wedge x_2 = 0$.

(3) Let, in addition, L be distributive and suppose that any two prime elements are incomparable (i.e. $\mathrm{Spec}\,L = \mathrm{Max}\,L$). Then $\mathrm{Spec}\,L$ is Hausdorff.
(Let $p_1, p_2 \in \mu L$, $p_1 \neq p_2$. Then $F_1 = L \setminus \downarrow p_1$ and $F_2 = L \setminus \downarrow p_2$ are Scott open filters. They are maximal. Hence by (2), there are $x_1 \in F_1$, $x_2 \in F_2$ such that $x_1 \wedge x_2 = 0$. We conclude
$S(x_1) \cap S(x_2) = \emptyset$, $p_1 \in S(x_1)$, $p_2 \in S(x_2)$.)

COROLLARY. (B.B.) $\boxed{X \text{ locally quasicompact, sober, } T_1,\ \ll \text{ multiplicative in } \mathcal{O}(X) \implies X\ T_2}$

(Proof. $L = \mathcal{O}(X)$ satisfies (3) and $X \cong \mathrm{Spec}\,\mathcal{O}(X)$.)