SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC: About polytopes of valuations on finite distributive lattices

REFERENCE:

[1] L. Geissinger, The face structure of a poset polytope, Proceedings of the Third Caribbean Conference on Combinatorics and Computings, University of the West Indies, Cave Hill, Barbados, 1981.

Let L be a finite distributive lattice. A mapping $v : L \longrightarrow \mathbb{R}$ is a valuation if $v(a+b) = v(a) + v(b) - v(ab)$ for all $a, b \in L$, and $v(0) = 0$. $V(L)$ denotes the real vector space of all valuations on L. The subset

$$M(L) = \{v \in V(L) \ : \ 0 \leq v \leq 1\}$$

is a convex polytope. In the sequel we shall verify the following conjecture of Geissinger [1]:

THEOREM A. <u>The extreme points of the convex polytope</u> $M(L)$ <u>are precisely the</u> 0-1 <u>valuations</u>.

Before proving it we shall formulate this statement in another way. Let P be a finite poset with n elements. By $L(P)$ we denote the distributive lattice of all lower sets of P, i.e., $A \in L(P)$ iff $A \subseteq P$ such that $y \leq x \in A$ always implies $y \in A$. (Recall that each finite distributive lattice L is isomorphic to $L(P)$ for the poset P of its prime elements.) A vector space isomorphism between $\mathbb{R}^P$ and $V(L(P))$ is given by the mapping

$$\Phi : h \longmapsto v_h : \begin{cases} L(P) \longrightarrow \mathbb{R} \\ A \longmapsto \sum_{p \in A} h(p) . \end{cases}$$

The convex polytope

$$M(P) = \{h \in \mathbb{R}^P : 0 \leq v_h \leq 1\}$$

is the image of $M(L(P))$ under ϕ^{-1}.

THEOREM B. _An element_ h _of the convex polytope_ $M(P)$ _is an extreme point if and only if_ $h = h_C$ _for some subchain_ C _of_ P, _where_ $C = \{p_o, p_1, \ldots, p_m\}$, $p_o < p_1 < \ldots < p_m$, _and_

$$h_C(p) = \begin{cases} 0 \ \text{for} \ p \in P - C, \\ (-1)^k \ \text{for} \ p = p_k. \end{cases}$$

Obviously, the h_C are exactly those elements of M(P) which are associated with a 0-1 valuation (cf. Geissinger [1; Proposition 2]). Therefore Theorem A and Theorem B are equivalent.

Proof of Theorem B. The if part has already been mentioned in [1]. In fact, obviously every 0-1 valuation is an extreme point even of $[0,1]^{L(P)}$. Conversely, let $e \in M(P)$ be an extreme point. We can assume that $e(p) \neq 0$ for all $p \in P$. Define for $i = 0, 1$

$L_i = \{A \in L(P) : v_e(A) = i\}$, $P_i = \bigcup \{A : A \in L_i\}$, and set

$A^* = \bigcap \{A : A \in L_1\}$.

Of course $\emptyset \in L_o$. As we shall see later, also L_1 is non-empty so that the above definitions of P_1 and A^* really make sense. L_o and L_1 are closed under non-empty unions and intersections; in particular P_i is the greatest element of L_i, and A^* is the smallest element of L_1.

Given a subset M of a vector space, let the rank $rk(M)$ of M be the dimension of the subspace generated by M. For $A \subseteq P$, the symbol δ_A refers to the characteristic function of A defined on P. Thus $v_h(A) = \delta_A \cdot h \quad (A \in L(P))$.

LEMMA 1. $rk\{\delta_A : A \in L_0 \cup L_1\} = n$.

Proof. Otherwise the intersection of all hyperplanes

$$H(A) = \{h \in \mathbb{R}^P : \delta_A \cdot (h-e) = 0\} \quad (A \in L_0 \cup L_1, \quad A \neq \emptyset)$$

contains a line $\{e + \lambda x_0 : \lambda \in \mathbb{R}\}$. For all $A \in L(P) - (L_0 \cup L_1)$ we have $0 < v_e(A) < 1$. Thus a continuity argument shows that for some $\varepsilon > 0$

$$\{e + \lambda x_0 : |\lambda| < \varepsilon\} \subseteq M(P),$$

a contradiction, since e is an extreme point. □

Here we insert a lemma of general character. The easy proof is left to the reader.

LEMMA 2. _Let_ X _be a finite set and_ K _a subset of the power set of_ X _which is closed under non-empty unions and arbitrary intersections (in particular_ $X \in K$_). Set_ $U_x = \bigcap \{U \in K : x \in U\}$, _and define_

$$x \leqq y \quad \text{iff} \quad U_x \subseteq U_y,$$

$$x \approx y \quad \text{iff} \quad U_x = U_y.$$

Then $\approx$ _is an equivalence relation on_ X, _and a partial ordering is given on_ $X/\approx$ _by setting_

$x/{\approx} \leq y/{\approx}$ _iff_ $x \leq y$.

The non-empty elements U _of_ K _are in a one-to-one correspondence with the non-empty lower sets of_ $X/{\approx}$ _via_

$$U \longmapsto \{u/{\approx} \; : \; u \in U\} .$$

Moreover we have $\mathrm{rk}\{\delta_U \in \mathbb{R}^X \; : \; U \in K\} = |X/{\approx}|$.

Let $\approx_i$ denote the equivalence relation on $X = P_i$ induced by $K = L_i$ in the sense of Lemma 2.

LEMMA 3. _Every_ $\approx_0$_-class contains at least two elements._

Proof. Let $p/{\approx_0}$ be a minimal element in $P_0/{\approx_0}$. Then by Lemma 2, $p/{\approx_0} \in L_0$, i. e. $v_e(p/{\approx_0}) = \sum_{q\approx p} e(q) = 0$. As $e(p) \neq 0$, we conclude that $|p/{\approx_0}| \geq 2$. So the assertion follows by induction over the height in $P_0/{\approx_0}$. □

Actually L_1 must be non-empty, because otherwise we obtain a contradiction in view of Lemma 1 and Lemma 3:

$$n = \mathrm{rk}\{\delta_A \; : \; A \in L_0 \cup L_1\} = \mathrm{rk}\{\delta_A \; : \; A \in L_0\} = |P_0/{\approx_0}| \leq \frac{n}{2}$$

LEMMA 4. _Every_ $\approx_1$_-class different from_ A^* _contains at least two elements._

Proof. A^* is the least element of L_1 , and $v_e(A^*) = \sum_{p \in A^*} e(p) = 1$. Therefore $\sum_{p \in A - A^*} e(p) = 0$ for all $A \in L_1$, and we can use the same argument as for Lemma 3 to prove the assertion. □

Because P_o or P_1 is a proper subset of P, we have

(1) $|P_o| + |P_1| \leqq 2n - 1$.

From Lemma 3 and Lemma 4 we conclude

(2) $2|P_o/\approx_o| \leq |P_o|$,

(3) $2(|P_1/\approx_1| - 1) + |A^*| \leq |P_1|$.

Further, by using these inequalities together with Lemma 1 and Lemma 2 we obtain

$$n = rk\{\delta_A : A \in L_o \cup L_1\} \leqq rk\{\delta_A : A \in L_o\} + rk\{\delta_A : A \in L_1\} =$$

$$= |P_o/\approx_o| + |P_1/\approx_1| \leq \frac{1}{2}(|P_o| + |P_1| - |A^*|) + 1 \leq n + \frac{1}{2}(1 - |A^*|).$$

Hence $|A^*| = 1$. On the other hand A^* contains all minimal elements of P (indeed, if $m \in P - A^*$ is minimal then $v_e(A^* \cup \{m\}) = 1 + e(m) > 1$). We infer that P has a least element p_o, $A^* = \{p_o\}$, and $e(p_o) = 1$. Now consider $P' = P - \{p_o\}$ and $e' = -e|_{P'}$. It follows that e' is an extreme point of $M(P')$. Thus P' has a least element, say p_1, and $e'(p_1) = -e(p_1) = 1$, etc. Finally we see that $P = \{p_o, p_1, \dots, p_{n-1}\}$ is a chain and $e(p_k) = (-1)^k$. This completes the proof of Theorem B.

Remark. Since it has been convenient in our present context, we have required that a valuation v satisfies $v(0) = 0$, a condition which is usually omitted. However, it is evident that this point does not touch Theorem A.